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Geometrical theory of uniform Cosserat media

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Abstract

A geometric description of generalized Cosserat media is presented in terms of non-holonomic frame bundles of second order. A non-holonomic $\bar{G}$-structure is constructed by using the smooth uniformity of the material and its integrability is proved to be equivalent to the homogeneity of the body. If the material enjoys global uniformity, the theory of linear connections in frame bundles permits to express the inhomogeneity by means of some tensor fields.

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1. Introduction

In Continuum Mechanics a material body $\mathcal{B}$ is represented by a three-dimensional differentiable manifold which can be covered with just one chart (see e.g. [65,66]). Such a chart $\Phi : \mathcal{B} \longrightarrow \mathbb{R}^3$ is called a configuration. It is customary to identify the body with any one of its configurations, $\Phi_0 : \mathcal{B} \longrightarrow \mathbb{R}^3$, called a reference configuration. A change of configuration $\kappa = \Phi \circ \Phi_0^{-1}$ is a deformation.

Experience accumulated over centuries of particular theories indicates that the mechanical behaviour of many material bodies is local, in the sense that the deformation evaluated outside an arbitrarily small neighbourhood of each point of $\mathcal{B}$ does not affect the material

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response at that point. The first derivative of the deformation, the deformation gradient $F = \nabla\kappa$, is sufficient for the description of the so-called simple materials. Here $\nabla\kappa$ denotes the derivative which coincides with the covariant derivative in the Euclidean context. Sometimes, we shall use this notation since it is usual in Continuum Mechanics.

A question of both theoretical and practical importance is the following: given a descriptor of the material behaviour of a simple material body as a function of position in the body, how can it be decided that all points of the body are made of the same material? Moreover, after having ascertained that this is the case, are there any inhomogeneities left which cannot be removed by a simple change of configuration? A geometric theory based on the properties of the material response function alone was developed by Noll [73] (see also [82–85]). A structurally based theory had been originally conceived by Kondo [55], Bilby [4], Kröner [56], Eshelby [48] and others, as the result of a limiting process starting from a defective crystalline structure (see also the books by Lardner [58] and Nabarro [72]). Following essentially Noll's and Wang's approach, the use of G-structure theory has refined the formulation and facilitated the derivation of specific results [31,44]. In fact, the presence of inhomogeneities, such as dislocations and disclinations, manifests itself through the lack of integrability of the associated G-structure.

In a sense, it may be said that the theory of inhomogeneities of simple elastic materials is fairly well established in terms of differential geometric constructs. However, many real materials are known to be non-simple. Granular solids, rocks, bone, animal blood, liquid crystals, composite materials, and many other materials which are common in nature cannot be faithfully modelled unless extra kinematic variables are taken into consideration [9,47]. The first theory of such *generalized media* was introduced by Eugène and François Cosserat between 1905 and 1910 [16]. We refer the reader to Pommaret [76] for an account of the life and works of the Cosserats. The Cosserats studied elastic curves, surfaces, and three-dimensional bodies to each point of which a family of vectors (or *directors*) is attached. More generally, a Cosserat continuum can be mathematically represented by an m-dimensional manifold $\mathcal{B}^m$ and a family of n vector fields $\{d_\alpha\}$ on $\mathcal{B}^m$. Many of the further developments of the theory can be found in Ericksen and Truesdell [46], Toupin [80,81], Maugin [67,69], Nowacki [74], Kröner [57], Antman [2] and the encyclopedical works of Truesdell and Toupin [82], Truesdell and Noll [83] and Eringen [47].

In spite of their importance, a complete theory of uniformity and homogeneity of generalized continua is not available. A correct definition of uniformity of micropolar and micromorphic media is given in [47], but without defining or exploiting the underlying geometrical apparatus so as to establish homogeneity conditions, as done by Noll and Wang for simple media.

The geometrical apparatus necessary to develop a rigorous theory has been available for some time. Actually, the notion of directors due to the Cosserats is closely related to the notion of the *repère mobile* (moving frame) due to Elie Cartan [10]. In fact, if $\mathcal{B}^m$ is a m-dimensional manifold, a set of m linearly independent vector fields on $\mathcal{B}^m$ is a moving frame. If we examine how the moving frame is deformed along curves on $\mathcal{B}^m$ we obtain the notion of covariant derivative and, hence, the notion of linear connection.

In the 1950s the second geometric ingredient for the theory was introduced. Charles Ehresmann (see [23–27] and references therein) formalized the notion of principal fibre bundle and studied several frame bundles associated in a natural way to an arbitrary manifold: non-holonomic, semi-holonomic and holonomic frame bundles. Connections of higher order were also introduced. The work of Eheresmann was continued by several of his students, Libermann [59–61,63,77], Yuen [86], and others [1,52,53,70,75]. We also refer to the recent book by Kolář et al. [54]. On the other hand, the notion of G-structure evolves from the works by Chern, Ehresmann, Bernard and Libermann (see [3,13]).

In this paper, we make use of these geometrical tools to study the uniformity and homogeneity of three-dimensional Cosserat media, by which we mean a three-dimensional continuum to each point of which three linearly independent tangent vectors are attached. Actually, we can interpret a generalized Cosserat continuum as a three-dimensional manifold plus its frame bundle. In fact, a linear frame at a point X of $\mathcal{B}$ is a basis of the tangent space $T_X\mathcal{B}$, i.e., a set of linearly independent tangent vectors at X. In order to determine the deformation of each tangent space it is necesssary to know how a basis of it is deformed. Thus, a configuration of a Cosserat medium would be an embedding of $F\mathcal{B}$ (the linear frame bundle of $\mathcal{B}$) into the linear frame bundle $F\mathbb{R}^3$ of $\mathbb{R}^3$ if we suppose that we deal with three-dimensional continua. The embedding of principal bundles induces an embedding between the base manifolds, $\mathcal{B}$ and $\mathbb{R}^3$. In this way, we recover the notion of configuration for simple materials. A deformation is a change of configuration, which is itself an isomorphism of principal bundles. If we fix an arbitrary configuration as a reference configuration, we obtain that a deformation is an embedding of $F\mathcal{B}$ into $F\mathbb{R}^3$.

The constitutive elastic law is now written as

$$W = W(j_X^1\tilde{\kappa}), \tag{1}$$

where $\tilde{\kappa}$ is the embedding and $j_X^1\,\tilde{\kappa}$ denotes the 1-jet of $\tilde{\kappa}$, or, in other words, the gradient of the deformation at a point X.

The constitutive equation (1) permits us to associate to each point of $\mathcal{B}$ an isotropy group (the group of material symmetries) as in the case of simple materials. If we assume that the medium enjoys uniformity then the isotropy groups at different points may be related by conjugation. If, further, the uniformity is smooth then we can construct a reduction of the non-holonomic frame bundle $\bar{F}^2\mathcal{B}$ of $\mathcal{B}$, i.e., a second-order non-holonomic $\bar{G}$-structure on $\mathcal{B}$, where $\bar{G}$ is a subgroup of the second-order non-holonomic group $\bar{G}^2(3)$. This kind of geometric structure was studied by Libermann, Oproiu, Yuen, Kolář [52,53,62,63,70,75] and others.

The associated $\bar{G}$-structure is obtained by using an algebraic-geometric object provided by the uniformity property, a Lie groupoid (see [64] for an excellent reference on Lie groupoids). In fact, the collection of material 1-jets of local isomorphisms connecting different points is a groupoid, its smoothness corresponding to the existence of local sections, which is equivalent to the Lie groupoid character.

As a particular case, second-order holonomic $\check{G}$-structures are obtained, which correspond just to materials of second grade. Thus, we obtain in a very natural way a general scheme including Cosserat media (non-holonomic) and materials of second grade (holonomic).

This geometric formulation also provides a natural extension of the continuous theories of inhomogeneities of Noll and Wang [8,68,73,83,85]. In fact, a non-holonomic parallelism induces a linear connection and also two ordinary parallelisms, which in turn define two linear connections on $\mathcal{B}$. The set of three linear connections defines two tensors: a torsion and a tensor of difference of connections. The local homogeneity of the material is equivalent to the vanishing of these inhomogeneity tensors. This result extends the one obtained by the authors for second grade materials [17–20] (see also [14,30–38]).

It should be noted that a more general geometrical model for so-called *continua with microstructure* [9,47] can be conceived as follows. A continuum with microstructure is a fibre bundle E over an m-dimensional continuum $\mathcal{B}$ with typical fibre F and projection $\pi : E \longrightarrow \mathcal{B}$. $\mathcal{B}$ is said to be the macromedium, F is the typical micromedium and the fibre $E_X = \pi^{-1}(X)$ is the micromedium attached at a point $X \in \mathcal{B}$. The simplest case is a trivial bundle $E = \mathcal{B} \times F \longrightarrow \mathcal{B}$. Slightly more complicated models are the ones of rods and shells; E is the normal bundle of a one-dimensional (resp. two-dimensional) continuum $\mathcal{B}$ into $\mathbb{R}^3$ (see [41–43]). Of course, our mathematical model for Cosserat media is a particular case of media with microstructure. The aim of this paper is to study Cosserat media, since they enjoy a richer geometrical structure. In [22,39,40] we have studied the general model of media with microstructure.

This paper is divided in two parts: a geometric part and the application to Cosserat media. In the first part, which consists of nine sections, we recall some definitions and results on linear frame bundles, non-holonomic, semi-holonomic and holonomic bundles of second-order as well as the corresponding structure groups. We notice that these concepts are known (but not extensively) to differential geometers. For this reason we shall explain them in some detail. On the other hand, we introduce a classification of the Lie subgroups of the second-order non-holonomic group $\bar{G}^2(n)$ as well as of the Lie subgroups of the second-order semi-holonomic and holonomic group $\hat{G}^2(n)$ and $G^2(n)$, respectively. Since there exists a complete classification of the Lie subgroups of the special linear group $\mathfrak{sl}(n, \mathbb{R})$ (see [83,85], for instance) we have a classification of the Lie subgroups of $\bar{G}^2(n)$ whose second projection onto $Gl(n, \mathbb{R})$ is a Lie subgroup of $\mathfrak{sl}(n, \mathbb{R})$. We shall use the formulation of jets throughout the paper. For the sake of completeness, we include a brief review of jets and Lie groupoids. The relationships between linear connections and invariant sections of the non-holonomic, semi-holonomic and holonomic frame bundle of second-order of a manifold M are established. Most of the results are known (see [15,49,51,52,70,78,79]), but some are new or presented in a new light (e.g. the notion of prolongability of a non-holonomic second-order G-structure).

The second part of the paper is devoted to the application of the results of the first part to a geometrical model for Cosserat media. We introduce the notions of configurations and deformations of Cosserat continua. The elastic constitutive equation is given and the uniformity property is established. The group of material symmetries is introduced. We also give a first geometrical characterization of the homogeneity in terms of the integrability of the associated non-holonomic second-order $\bar{G}$-structure. This non-holonomic second-order $\bar{G}$-structure is obtained by introducing a crystal reference. The behaviour of the fields of uniformity under the changes of crystal reference and reference configuration are carefully studied. Then we study the case of a Cosserat continuum enjoying global smooth uniformity. Finally, particular cases of Cosserat continua are studied and their integrability is characterized by means of some inhomogeneity tensors.

Part I. Geometric background on frame bundles

2. Principal bundles

Let M be a manifold and G a Lie group. Roughly speaking, a *principal bundle* P over M with structure group G is obtained attaching a copy of G to each point of M. More precisely, P is a manifold on which G acts by the right and satisfying the following conditions:

(i) The action of G is free, i.e., $ua = R_a(u) = u$, $u \in P$, implies $a = e$, where e is the identity of G.

(ii) $M = P/G$, i.e., M is the quotient space of P by the equivalence relation induced by G. In other words, M is the space of orbits. Moreover, the canonical projection $\pi : P \longrightarrow M$ is differentiable.

(iii) P is locally trivial, i.e., P is locally a product $U \times G$, where U is an open set of M. More precisely, there exists a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times G$, such that $\Phi(u) = (\pi(u), \varphi(u))$, where the mapping $\varphi : \pi^{-1}(U) \longrightarrow G$ satisfies $\varphi(ua) = \varphi(u)a$ for all $u \in \pi^{-1}(U)$, $a \in G$.

A principal bundle will be denoted by $P(M, G)$, or simply $\pi : P \longrightarrow M$ if there is no ambiguity as to the structure group G. P is called the *total space*, M the *base space*, G the *structure group*, and π the *projection*. The closed submanifold $\pi^{-1}(x)$, $x \in M$ will be called the *fibre over* x. For a point $u \in P$, we have $\pi^{-1}(x) = uG$, where $\pi(u) = x$, and uG will be called the *fibre trough* u. Every fibre is diffeomorphic to G, but this diffeomorphism depends on the choice of the trivialization.

Given a manifold M and a Lie group G the product manifold $M \times G$ is a principal bundle over M with projection $pr_1 : M \times G \longrightarrow M$ and structure group G, the action given by $(x, a)b = (x, ab)$. $M \times G$ is called a *trivial* principal bundle.

A *homomorphism* of a principal bundle $P'(M', G')$ into another principal bundle $P(M, G)$ consists of a differentiable mapping $\Phi : P' \longrightarrow P$ and a Lie group homomorphism $\varphi : G' \longrightarrow G$ such that $\Phi(u'a') = \Phi(u')\varphi(a')$ for all $u' \in P'$ and $a' \in G'$. Hence, Φ maps fibres into fibres and it induces a differentiable mapping $\phi : M' \longrightarrow M$ by $\phi(x') = \pi(\Phi(u'))$, where u' is an arbitrary point over x'. A homomorphism $\Phi : P' \longrightarrow P$ is called an *embedding* if $\phi : M' \longrightarrow M$ is an embedding and if $\varphi : G' \longrightarrow G$ is injective. In such a case, we can identify P' with $\Phi(P')$, G' with $\varphi(G')$ and M' with $\phi(M')$ and P' is said to be a *subbundle* of P. If $M' = M$ and $\phi = id_M$, P' is called a *reduced subbundle* and we also say that G *reduces* to the subgroup G'.

A homomorphism $\Phi : P' \longrightarrow P$ is called an *isomorphism* if there exists a homomorphism of principal bundles $\Psi : P \longrightarrow P'$ such that $\Psi \circ \Phi = id_{P'}$ and $\Phi \circ \Psi = id_P$.

3. Frame bundles

Let M be an n-dimensional differentiable manifold. A linear frame at the point x is a linear isomorphism $z : \mathbb{R}^n \longrightarrow T_xM$. Alternatively, z may be viewed as an ordered basis $\{z_1, \dots, z_n\}$ of T_xM, with $z_i = z(r_i)$, where $\{r_1, \dots, r_n\}$ is the canonical basis of $\mathbb{R}^n$. There exists a third way to interpret a linear frame by using the theory of jets. Indeed, a linear frame z at x may be considered as the 1-jet $j^1_{0,x}\phi$ of a local diffeomorphism ϕ from an open neighbourhood of 0 in $\mathbb{R}^n$ onto an open neighbourhood of x in M such that $\phi(0) = x$. We have $z = \mathrm{d}\phi(0) : \mathbb{R}^n \longrightarrow T_xM$.

We denote by FM the set of all linear frames at all the points of M. As is well-known, FM is a principal bundle over M with structure group $Gl(n, \mathbb{R})$ and projection $\pi_M : FM \longrightarrow M$ defined by $\pi_M(j^1_{0,x}\phi) = \phi(0) = x$. We denote by e_1 the element $j^1_{0,0}id_{\mathbb{R}^n} \in F\mathbb{R}^n$. If $\psi : N \longrightarrow M$ is a local diffeomorphism from an n-dimensional manifold N into another n-dimensional manifold M, we denote by $F\psi : FN \longrightarrow FM$ the local isomorphism induced from ψ, and defined by

$$F\psi(j^1_{0,\phi(0)}\phi) = j^1_{0,\psi(\phi(0))}(\psi \circ \phi).$$

Let $\Psi : F\mathbb{R}^n \longrightarrow FM$ be a local isomorphism of principal bundles such that its domain contains e_1 and the induced isomorphism on Lie groups is the identity. Then we have $\Psi(za) = \Psi(z)a$ for all $z \in F\mathbb{R}^n$ and for all $a \in Gl(n, \mathbb{R})$. We denote by $\psi : \mathbb{R}^n \longrightarrow M$ the local diffeomorphism induced by Ψ. We recall that $\psi \circ \pi_{\mathbb{R}^n} = \pi_M \circ \Psi$. The collection of all 1-jets $j^1_{e_1,\Psi(e_1)}\Psi$ is a manifold which will be denoted by $\bar{F}^2M$. Of course, $j^1_{e_1,\Psi(e_1)}\Psi$ can be identified with a linear frame at the point $\Psi(e_1)$ since $\mathrm{d}\Psi(e_1) : T_{e_1}(F\mathbb{R}^n) \cong \mathbb{R}^{n+n^2} \longrightarrow T_{\Psi(e_1)}(FM)$ is a linear isomorphism, and we have $\bar{F}^2M \subset F(FM)$. There are two canonical projections $\bar{\pi}^2_1 : \bar{F}^2M \longrightarrow FM$ and $\bar{\pi}^2 : \bar{F}^2M \longrightarrow M$ given by $\bar{\pi}^2_1(j^1_{e_1,\Psi(e_1)}\Psi) = \Psi(e_1)$ and $\bar{\pi}^2(j^1_{e_1,\Psi(e_1)}\Psi) = \psi(0)$, respectively. Of course, we have $\bar{\pi}^2 = \pi_M \circ \bar{\pi}^2_1$. It can be shown that $\bar{F}^2M$ is a principal bundle over FM with canonical projection $\bar{\pi}^2_1$ and structure group $\bar{G}^2_1(n)$ consisting of all 1-jets of local isomorphisms of $F\mathbb{R}^n$ into $F\mathbb{R}^n$ with source and target e_1. Hence, $\bar{G}^2_1(n)$ is a Lie subgroup of $Gl(n+n^2, \mathbb{R})$ acting on $\bar{F}^2M$ by composition of jets.

We also have that $\bar{F}^2M$ is a principal bundle over M with canonical projection $\bar{\pi}^2$ and structure group $\bar{G}^2(n)$. The group $\bar{G}^2(n)$ is the fibre of $\bar{F}^2(\mathbb{R}^n)$ over $0 \in \mathbb{R}^n$, i.e., $\bar{G}^2(n) = (\bar{\pi}^2)^{-1}(0)$.

An alternative description of the Lie group $\bar{G}^2(n)$ is the following. It consists of all 1-jets $j^1_{e_1,\Psi(e_1)}\Psi$ of local isomorphisms $\Psi : F\mathbb{R}^n \longrightarrow F\mathbb{R}^n$ such that the induced map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ maps 0 into 0. The multiplication is given by composition of jets:

$$(j^1_{e_1\Psi_1(e_1)}\Psi_1)\,(j^1_{e_1,\Psi_2(e_1)}\Psi_2) = j^1_{e_1,\Psi_1(\Psi_2(e_1))}(\Psi_1 \circ \Psi_2).$$

The action of $\bar{G}^2(n)$ on $\bar{F}^2(M)$ is also given by composition of jets. The bundle $\bar{F}^2M$ will be called the *non-holonomic frame bundle of second order* and its elements will be called *non-holonomic frames of second order*. Notice that there exists a canonical isomorphism $\bar{F}^2\mathbb{R}^n \cong \mathbb{R}^n \times \bar{G}^2(n)$. In fact, define a global section $s : \mathbb{R}^n \longrightarrow \bar{F}^2\mathbb{R}^n$ as follows:

$$s(x) = j^1_{e_1,\Psi_x(e_1)}\Psi_x,$$

where $\Psi_x(s^i, s^i_j) = (r^i + s^i, s^i_j)$, $x = (r^i) \in \mathbb{R}^n$, and (s^i, s^i_j) being the canonical coordinates on $F\mathbb{R}^n$. So, a non-holonomic frame of second-order u at a point $x \in \mathbb{R}^n$ may be written in a unique way as $u = s(x)g$, where $g \in \bar{G}^2(n)$. We have thus obtained a principal bundle isomorfism $\bar{F}^2\mathbb{R}^n \cong \mathbb{R}^n \times \bar{G}^2(n)$. Now, if $\bar{G}$ is a Lie subgroup of $\bar{G}^2(n)$, we can transport $\mathbb{R}^n \times \bar{G}$ by this isomorphism to obtain a $\bar{G}$-reduction of $\bar{F}^2(\mathbb{R}^n)$.

Definition 3.1. Let $\bar{G}$ be a Lie subgroup of $\bar{G}^2(n)$. A $\bar{G}$-reduction $\bar{\omega}_{\bar{G}}(M)$ of $\bar{F}^2(M)$ to the group $\bar{G}$ will be called a *second-order non-holonomic* $\bar{G}$-structure.

Hence, the $\bar{G}$-reduction of $\bar{F}^2\mathbb{R}^n$ obtained above is a second-order non-holonomic $\bar{G}$-structure on $\mathbb{R}^n$ which will be called the *standard flat* (or integrable) second-order non-holonomic $\bar{G}$-structure.

Definition 3.2. A second-order non-holonomic $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(M)$ on M will be called *integrable* if it is locally isomorphic to the standard flat $\bar{G}$-structure on $\mathbb{R}^n$.

Notice that an integrable second-order non-holonomic $\bar{G}$-structure is not necessarily holonomic (see Definition 3.7). We shall give a weaker notion of integrability in Section 7.

A second-order non-holonomic trivial structure is called a *non-holonomic parallelism of second order*. Let us recall that a linear parallelism on a manifold M is just a global section of the linear frame bundle FM, or, alternatively, a usual $\{1\}$-structure. A direct computation shows that a non-holonomic parallelism of second order is, in fact, equivalent to give a global smooth section of $\bar{\pi}^2 : \bar{F}^2M \longrightarrow M$.

Next, we shall describe two particular sub-bundles of $\bar{F}^2M$. Consider the second-order non-holonomic frames $j^1_{e_1,\Psi(e_1)}\Psi$ such that Ψ is admissible, i.e., $\Psi(e_1) = j^1_{0,\psi(0)}\psi$. Such a frame will be called a *semi-holonomic frame of second order* and the set $\hat{F}^2M$ of all these frames is called the *second-order semi-holonomic frame bundle* of M. We have canonical projections $\hat{\pi}^2_1 : \hat{F}^2M \longrightarrow FM$ and $\hat{\pi}^2 : \hat{F}^2M \longrightarrow M$, given by the restrictions of $\bar{\pi}^2_1$ and $\bar{\pi}^2$, respectively. As in the case of second-order non-holonomic frames we have that $\hat{F}^2M$ is a principal bundle over FM with canonical projection $\hat{\pi}^2_1$ and structure group $\hat{G}^2_1(n)$ consisting of the 1-jets of all admissible local isomorphisms of $F\mathbb{R}^n$ into $F\mathbb{R}^n$ with source and target e_1. As above, we deduce that $\hat{G}^2_1(n)$ is a Lie subgroup of $Gl(n+n^2, \mathbb{R})$ acting on $\hat{F}^2M$ by composition of jets.

$\hat{F}^2M$ is also a principal bundle over M with canonical projection $\hat{\pi}^2$ and structure group $\hat{G}^2(n)$. The structure group $\hat{G}^2(n)$ is defined by $\hat{G}^2(n) = (\hat{\pi}^2)^{-1}(0)$.

An alternative description of the Lie group $\hat{G}^2(n)$ is the following. It consists of the 1-jets $j^1_{e_1,\Psi(e_1)}\Psi$ of all admissible local isomorphisms $\Psi : F\mathbb{R}^n \longrightarrow F\mathbb{R}^n$ such that the induced map $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ maps 0 into 0. The multiplication and right action are given again by composition of jets. It is easy to prove that $\hat{\pi}^2_1 : \hat{F}^2M \longrightarrow FM$ (resp. $\hat{\pi}^2 : \hat{F}^2M \longrightarrow M$) is a principal sub-bundle of $\bar{\pi}^2_1 : \bar{F}^2M \longrightarrow FM$ (resp. $\bar{\pi}^2 : \bar{F}^2M \longrightarrow M$). Notice that there exists a canonical isomorphism $\hat{F}^2\mathbb{R}^n \cong \mathbb{R}^n \times \hat{G}^2(n)$. If $\hat{G}$ is a Lie subgroup of $\hat{G}^2(n)$, then we obtain a $\hat{G}$-reduction of $\hat{F}^2\mathbb{R}^n$ which is isomorphic with $\mathbb{R}^n \times \hat{G}$.

Definition 3.3. Let $\hat{G}$ be a Lie subgroup of $\hat{G}^2(n)$. A $\hat{G}$-reduction $\hat{\omega}_{\hat{G}}(M)$ of $\hat{F}^2(M)$ to the subgroup $\hat{G}$ will be called a *second-order semi-holonomic* $\hat{G}$-structure.

Hence, the canonical $\hat{G}$-reduction of $\hat{F}^2\mathbb{R}^n$ defined above is a second-order semi-holonomic $\hat{G}$-structure on $\mathbb{R}^n$ and it is called the *standard flat* (or integrable) second-order semi-holonomic $\hat{G}$-structure.

Definition 3.4. A second-order semi-holonomic $\hat{G}$-structure $\hat{\omega}_{\hat{G}}(M)$ on M will be called *integrable* if it is locally isomorphic to the standard flat $\hat{G}$-structure $\mathbb{R}^n \times \hat{G}$.

A second-order semi-holonomic trivial structure is called a *semi-holonomic parallelism of second-order*. A semi-holonomic parallelism of second order is, in fact, a global smooth section of $\hat{\pi}^2 : \hat{F}^2M \longrightarrow M$.

Remark 3.5. Notice that there exists a canonical projection $\tilde{\pi}^2_1 : \bar{F}^2M \longrightarrow FM$ defined by $\tilde{\pi}^2_1(j^1_{e_1,\Psi(e_1)}\Psi) = j^1_{0,\psi(0)}\psi$. Indeed, $\tilde{\pi}^2_1$ is nothing but the restriction of the canonical projection $\pi_{FM} : F(FM) \longrightarrow FM$ to $\bar{F}^2M$, and so it is a principal bundle homomorphism. It directly follows from the definitions that a second-order non-holonomic frame $\bar{z}$ is semi-holonomic if and only if $\bar{\pi}^2_1(\bar{z}) = \tilde{\pi}^2_1(\bar{z})$.

Finally, we shall introduce a new principal sub-bundle of $\bar{F}^2M$. Consider the second-order non-holonomic frames $j^1_{e_1,\Psi(e_1)}\Psi$ of M such that $\Psi = F\psi$. Hence, Ψ is admissible. Such a frame will be called a *holonomic frame of second order* and the set F^2M of all these frames is called the *second-order holonomic frame bundle* of M, or, simply the *second-order frame bundle* of M. We get canonical projections $\pi^2_1 : F^2M \longrightarrow FM$ and $\pi^2 : F^2M \longrightarrow M$. We have that F^2M is a principal bundle over FM with structure group $G^2_1(n)$ consisting of all 1-jets of local isomorphisms of the form $F\psi$, where $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a local diffeomorphism such that $\psi(0) = 0$. Hence, $G^2(n)$ is a Lie subgroup of $Gl(n+n^2, \mathbb{R})$ acting on F^2M by composition of jets.

F^2M is also a principal bundle over M with canonical projection π^2 and structure group $G^2(n)$. The structure group $G^2(n)$ is defined by $G^2(n) = (\pi^2)^{-1}(0)$.

Alternatively, we can easily see that the Lie group $G^2(n)$ consists of all 1-jets $j^1_{e_1,\Psi(e_1)}\Psi$ of local isomorphisms $\Psi : F\mathbb{R}^n \longrightarrow F\mathbb{R}^n$ of the form $\Psi = F\psi$, $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The multiplication and right action are given again by composition of jets. We deduce that $\pi^2_1 : F^2M \longrightarrow FM$ (resp. $\pi^2 : F^2M \longrightarrow M$) is a principal sub-bundle of $\hat{\pi}^2_1 : \hat{F}^2M \longrightarrow FM$ (resp. $\hat{\pi}^2 : \hat{F}^2M \longrightarrow M$). Notice that $F^2\mathbb{R}^n \cong \mathbb{R}^n \times G^2(n)$. If G is a Lie subgroup of $G^2(n)$, then we obtain a G-reduction of $F^2\mathbb{R}^n$ which is isomorphic with $\mathbb{R}^n \times G$.

Definition 3.6. Let G be a Lie subgroup of $G^2(n)$. A G-reduction $\omega_G(M)$ of $F^2(M)$ to the subgroup G will be called a *second-order holonomic* G-structure (or *second-order* G-structure, for the sake of simplicity).

Hence, the canonical G-reduction of $F^2\mathbb{R}^n$ defined above is a second-order G-structure on $\mathbb{R}^n$ and it is called the *standard flat* (or integrable) second-order G-structure.

Definition 3.7. A second-order G-structure $\omega_G(M)$ on M will be called *integrable* if it is locally isomorphic to the standard flat G-structure $\mathbb{R}^n \times G$.

A second-order holonomic trivial structure is called a *holonomic parallelism of second order*. A holonomic parallelism of second order is, in fact, a global smooth section of $\pi^2 : F^2M \longrightarrow M$.

A direct computation shows that an integrable non-holonomic parallelism of second order is in fact holonomic.

Summarizing we have the following two sequences of Lie subgroups:

$$G^2(n) \subset \hat{G}^2(n) \subset \bar{G}^2(n) \subset Gl(n, \mathbb{R}) \times Gl(n + n^2, \mathbb{R}),$$
$$G^2_1(n) \subset \hat{G}^2_1(n) \subset \bar{G}^2_1(n) \subset Gl(n + n^2, \mathbb{R}),$$

and the following two sequences of principal bundles:

$$F^2M \subset \hat{F}^2M \subset \bar{F}^2M \subset F(FM)$$

over FM, and

$$F^2M \subset \hat{F}^2M \subset \bar{F}^2M$$

over M.

Remark 3.8. There exists an alternative definition of non-holonomic frames (see [75]). Consider a differentiable mapping $\phi : U \longrightarrow FM$ defined on some open neighbourhood of 0 in $\mathbb{R}^n$ such that $\pi_M \circ \phi : U \longrightarrow M$ is a diffeomorphism. Then the 1-jet $j^1_{0,\phi(0)}\phi$ is a non-holonomic frame of second order at $x = \pi_M(\phi(0))$. In fact, given ϕ we define a local principal bundle isomorphism $\Phi : F\mathbb{R}^n \longrightarrow FM$ over U by putting $\Phi(r, R) = \phi(r)R$, where $r = (r^i) \in \mathbb{R}^n$ and $R = (R^i_j) \in Gl(n, \mathbb{R})$. Thus, $j^1_{e_1,\Phi(e_1)}\Phi$ defines a non-holonomic frame at x, and a fortiori a linear frame at $\phi(0) \in FM$. A simple computation shows that every non-holonomic frame of second-order may be obtained in this way. In this formulation, the condition of semi-holonomicity is equivalent to the following one:

$$\phi(0) = j^1_{0,x}(\pi_M \circ \phi).$$

The holonomicity condition is given by

$$\phi(r) = j^1_{r,(\pi_M \circ \phi)(r)}(\pi_M \circ \phi) \quad \text{for all } r \in U.$$

4. Local descriptions

All the notions introduced in the previous section are of global nature. However, we shall now introduce local coordinates in our picture. In fact, the local description of all these bundles will be useful in our study.

Let (x^i) be a local coordinate system defined on some open subset U on M. We denote by FU the open subset of FM defined by $FU = (\pi_M)^{-1}(U)$. Notice that our notation is consistent, since FU is in fact the linear frame bundle of U. The following identities and notations are the obvious ones:

$$\bar{F}^2U = (\bar{\pi}^2)^{-1}(U), \qquad \hat{F}^2U = (\hat{\pi}^2)^{-1}(U),$$
$$F^2U = (\pi^2)^{-1}(U), \qquad F(FU) = (\pi_{FM})^{-1}(U).$$

The induced coordinates are denoted as follows:

$$\begin{aligned}
FU:&\quad (x^i, x^i_j),\\
F(FU):&\quad (x^i, x^i_j; x^i_{,j}, x^i_{,jk}, x^i_{j,k}, x^i_{j,kl}),\\
\bar{F}^2U:&\quad (x^i, x^i_j; x^i_{,j}, x^i_{,jk} = 0, x^i_{j,k}, x^i_{j,kl} = x^i_k\delta_{jl}),\\
\hat{F}^2U:&\quad (x^i, x^i_j; x^i_{,j} = x^i_j, x^i_{,jk} = 0, x^i_{j,k}, x^i_{j,kl} = x^i_k\delta_{jl}),\\
F^2U:&\quad (x^i, x^i_j; x^i_{,j} = x^i_j, x^i_{,jk} = 0, x^i_{j,k}, x^i_{j,kl} = x^i_k\delta_{jl}),\quad x^i_{j,k} = x^i_{k,j}.
\end{aligned}$$

For sake of simplicity, the local coordinates on $\bar{F}^2M$, $\tilde{F}^2M$ and F^2M will be written as follows:

$$\bar{F}^2U:\quad (x^i, x^i_j, x^i_{,j}, x^i_{j,k}), \qquad \hat{F}^2U: (x^i, x^i_j, x^i_{j,k}),$$
$$F^2U:\quad (x^i, x^i_j, x^i_{jk}),\quad x^i_{jk} = x^i_{kj},$$

and the canonical projections may now be written as follows:

$$\pi_{FM}(x^i, x^i_j; x^i_{,j}, x^i_{,jk}, x^i_{j,k}, x^i_{j,kl}) = (x^i, x^i_j), \qquad \bar{\pi}^2_1(x^i, x^i_j, x^i_{,j}, x^i_{j,k}) = (x^i, x^i_j),$$
$$\bar{\pi}^2(x^i, x^i_j, x^i_{,j}, x^i_{j,k}) = (x^i), \qquad \bar{\pi}^2_1(x^i, x^i_j, x^i_{,j}, x^i_{j,k}) = (x^i, x^i_{,j}),$$
$$\hat{\pi}^2_1(x^i, x^i_j, x^i_{j,k}) = (x^i, x^i_j), \qquad \hat{\pi}^2(x^i, x^i_j, x^i_{j,k}) = (x^i),$$
$$\pi^2_1(x^i, x^i_j, x^i_{jk}) = (x^i, x^i_j), \qquad \pi^2(x^i, x^i_j, x^i_{jk}) = (x^i), \qquad \pi_M(x^i, x^i_j) = (x^i).$$

Using these notations we can write the elements of the different Lie groups $\bar{G}^2(n)$, $\hat{G}^2(n)$, $G^2(n)$, $\bar{G}^2_1(n)$, $\hat{G}^2_1(n)$ and $G^2_1(n)$ as follows:

$$\begin{aligned}
Gl(n+n^2, \mathbb{R}):&\quad A = (A^i_{,j}, A^i_{,jk}, A^i_{j,k}, A^i_{j,kl}),\\
\bar{G}^2(n):&\quad A = (A^i_j, A^i_{,j}, A^i_{j,k}),\\
\hat{G}^2(n):&\quad A = (A^i_j, A^i_{j,k}),\\
G^2(n):&\quad A = (A^i_j, A^i_{jk}),\quad A^i_{jk} = A^i_{kj},\\
\bar{G}^2_1(n):&\quad A = (A^i_{,j}, A^i_{j,k}),\\
\hat{G}^2_1(n):&\quad A = (A^i_{j,k}),\\
G^2_1(n):&\quad A = (A^i_{jk}),\quad A^i_{jk} = A^i_{kj},
\end{aligned}$$

and the corresponding multiplications are then given by

$$
\begin{aligned}
&\bar{G}^2(n): && (AB)^i_j = A^i_k B^k_j,\ (AB)^i_{,j} = A^i_{,k} B^k_{,j},\ (AB)^i_{j,k} = A^i_r B^r_{j,k} + A^i_{r,s} B^r_j B^s_{,k},\\
&\hat{G}^2(n): && (AB)^i_j = A^i_k B^k_j,\ (AB)^i_{j,k} = A^i_r B^r_{j,k} + A^i_{r,s} B^r_j B^s_k,\\
&G^2(n): && (AB)^i_j = A^i_k B^k_j,\ (AB)^i_{jk} = A^i_r B^r_{jk} + A^i_{rs} B^r_j B^s_k,\\
&\bar{G}^2_1(n): && (AB)^i_{,j} = A^i_{,k} B^k_{,j},\ (AB)^i_{j,k} = B^i_{j,k} + A^i_{j,s} B^s_{,k},\\
&\hat{G}^2_1(n): && (AB)^i_{j,k} = B^i_{j,k} + A^i_{j,k},\\
&G^2_1(n): && (AB)^i_{jk} = B^i_{jk} + A^i_{jk}.
\end{aligned}
$$

From Definitions 3.2, 3.4 and 3.7 and the above local expressions, we directly obtain the following:

Proposition 4.1. *A second-order non-holonomic (resp. semi-holonomic, holonomic) $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(M)$ (resp. $\hat{G}$-structure $\hat{\omega}_{\hat{G}}(M)$, G-structure $\omega_G(M)$) on M is integrable if and only if for any point $x \in M$ there exists a local coordinate neighbourhood U with local coordinates (x^i) such that the local section $Q(x^i) = (x^i, 1, 1, 0)$ takes values into $\bar{\omega}_{\bar{G}}(M)$ (resp. $\hat{\omega}_{\hat{G}}(M)$, $\omega_G(M)$).*

Denote by $B^2(n)$ the vector space of the bilinear mappings from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$. Hence, there exists a canonical inclusion $j : Gl(n, \mathbb{R}) \longrightarrow \bar{G}^2(n) = Gl(n, \mathbb{R}) \times Gl(n, \mathbb{R}) \times B^2(n)$ defined by $j(A) = (A, A, 0)$. Notice that j is in fact a Lie group homomorphism since $j(AB) = j(A)j(B)$.

Denote by $S^2(n) \subset B^2(n)$ the vector subspace of symmetric bilinear mappings. We have a canonical inclusion (denoted by the same letter) $j : Gl(n, \mathbb{R}) \longrightarrow G^2(n) = Gl(n, \mathbb{R}) \times S^2(n)$ defined by $j(A) = (A, 0)$. In fact, $j : Gl(n, \mathbb{R}) \longrightarrow G^2(n)$ is the restriction of $j : Gl(n, \mathbb{R}) \longrightarrow \bar{G}^2(n)$ taking into account that $G^2(n)$ (and $\hat{G}^2(n)$ too) may be viewed as a Lie subgroup of $\bar{G}^2(n)$ by identifying (A, α) with (A, A, α).

5. More about the Lie groups $\bar{G}^2(n)$, $\hat{G}^2(n)$ and $G^2(n)$

In this section we shall describe in an alternative way the Lie groups $\bar{G}^2(n)$, $\hat{G}^2(n)$ and $G^2(n)$. We shall also give a classification of their Lie subgroups.

First of all, let us recall the definition of $\bar{G}^2(n)$. A typical element is a 1-jet $j^1_{e_1, \Psi(e_1)} \Psi$, such that $\psi(0) = 0$. In local coordinates we have $\Psi(r^i, r^i_j) = (\psi^i(r^a), \Psi^i_k(r^a, 1) r^k_j)$, where (r^a, r^a_b) denotes the canonical coordinates in $F\mathbb{R}^n$. Hence, $j^1_{e_1, \Psi(e_1)} \Psi$ defines a triple (A, A', α) by taking

$$
A = (A^i_j), \qquad A^i_j = \Psi^i_j(0, 1), \qquad A' = (A'^i_{,j}),
$$

$$
A'^i_{,j} = \frac{\partial \psi^i}{\partial r^j}(0), \qquad \alpha = (\alpha^i_{jk}), \qquad \alpha^i_{jk} = \frac{\partial \Psi^i_j}{\partial r^k}(0).
$$

Thus, we can interpret A and A' as linear automorphisms of $\mathbb{R}^n$. In fact, A is the linear automorphism of $\mathbb{R}^n$ defined by $\Psi(0, 1)$, which is a linear frame at $0 \in \mathbb{R}^n$ taking into account the identification $T_0\mathbb{R}^n \cong \mathbb{R}^n$. On the other hand, A' is the linear isomorphism $d\psi(0): \mathbb{R}^n \cong T_0\mathbb{R}^n \longrightarrow \mathbb{R}^n \cong T_0\mathbb{R}^n$. Finally, α is the bilinear mapping $\alpha: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\alpha(u, v) = \tilde{\alpha}(u)(v)$, where $\tilde{\alpha}: \mathbb{R}^n \longrightarrow End(\mathbb{R}^n)$ is the differential of the mapping Ψ_j^i evaluated at 0.

If we put

$$A(r_j) = A^i_j r_i, \qquad A'(r_j) = A'^i_{.j}\, r_i, \qquad \alpha(r_j, r_k) = \alpha^i_{jk}\, r_i,$$

we deduce that the group $\bar{G}^2(n)$ may be identified with the product $Gl(n, \mathbb{R}) \times Gl(n, \mathbb{R}) \times B^2(n)$. The multiplication is now given by

$$(A, A', \alpha)\ (B, B', \beta) = (AB, A'B', A\beta + \alpha(B, B')), \tag{2}$$

where

$$A\beta(u, v) = A(\beta(u, v)), \qquad \alpha(B, B')(u, v) = \alpha(Bu, B'v)$$

for all $u, v \in \mathbb{R}^n$. Notice that this multiplication is just the one given by Eringen (see [47], and [1,15] for the holonomic case).

The neutral element is $(1, 1, 0)$ and the inverse element of an arbitrary element (A, A', α) is $(A^{-1}, A'^{-1}, -A^{-1}\alpha(A^{-1}, A'^{-1}))$.

Since $\bar{G}^2(n) \cong Gl(n, \mathbb{R}) \times Gl(n, \mathbb{R}) \times B^2(n)$, we have three canonical projections denoted by $pr_1: \bar{G}^2(n) \longrightarrow Gl(n, \mathbb{R})$, $pr_2: \bar{G}^2(n) \longrightarrow Gl(n, \mathbb{R})$ and $pr_3: \bar{G}^2(n) \longrightarrow B^2(n)$.

From (2) we deduce that pr_1 and pr_2 are Lie group homomorphisms. In fact, pr_1 (resp. pr_2) is induced by the canonical projection $\bar{\pi}^2_1$ (resp. $\tilde{\pi}^2_1$). However, pr_3 is not a Lie group homomorphism. Therefore, an arbitrary Lie subgroup $\bar{G}$ of $\bar{G}^2(n)$ may be written as follows: $\bar{G} = (G_1, G_2, \Sigma)$, where $G_1 = pr_1(\bar{G})$ and $G_2 = pr_2(\bar{G})$ are Lie subgroups of $Gl(n, \mathbb{R})$, and $\Sigma = pr_3(\bar{G})$ is a subset of $B^2(n)$. Given two arbitrary elements $A \in G_1$ and $A' \in G_2$ we denote by $\Sigma_{(A,A')}$ the subset of $B^2(n)$ defined by

$$\Sigma_{(A,A')} = \{\alpha \in B^2(n) \mid (A, A', \alpha) \in \bar{G}\}.$$

It is easy to check that $\Sigma_{(1,1)}$ is an additive subgroup of $B^2(n)$ and $(1, 1, \Sigma_{(1,1)})$ is a Lie subgroup of $\bar{G}^2(n)$.

Proposition 5.1. *For an arbitrary element $(A, A') \in G_1 \times G_2$ there exists a one-to-one correspondence between $\Sigma_{(1,1)}$ and $\Sigma_{(A,A')}$.*

Proof. Let (A, A', α_0) be an arbitrary element of $\bar{G}$. If $(1, 1, \tau) \in (1, 1, \Sigma_{(1,1)})$ we deduce that $(A, A', \alpha_0)(1, 1, \tau) = (A, A', A\tau + \alpha_0)$. Hence, we have obtained a mapping $\phi: \Sigma_{(1,1)} \longrightarrow \Sigma_{(A,A')}$ defined by $\phi(\tau) = A\tau + \alpha_0$.

Conversely, since the product $(A, A', \alpha_0)^{-1}(A, A', \alpha) = (1, 1, A^{-1}(\alpha - \alpha_0))$ belongs to $(1, 1, \Sigma_{(1,1)})$ for each $(A, A', \alpha) \in \bar{G}$ then we obtain a mapping $\psi: \Sigma_{(A,A')} \longrightarrow \Sigma_{(1,1)}$ defined by $\psi(\alpha) = A^{-1}(\alpha - \alpha_0)$.

A direct computation shows that $\psi \circ \phi = id$ and $\phi \circ \psi = id$. □

Consider the second-order semi-holonomic and holonomic groups $\hat{G}^2(n)$ and $G^2(n)$, respectively. In the first case, we have an identification $\hat{G}^2(n) \cong Gl(n,\mathbb{R}) \times B^2(n)$, since $A' = A$. In the second case, we have an identification $G^2(n) \cong Gl(n,\mathbb{R}) \times S^2(n)$. The multiplication (2) reads now as

$$(A,\alpha)\,(B,\beta) = (AB,\, A\beta + \alpha(B,B)). \tag{3}$$

The neutral element is $(1,0)$ and the inverse element of an arbitrary element (A,α) is $(A^{-1}, -A^{-1}\alpha(A^{-1}, A^{-1}))$. We denote by $pr_1 : \hat{G}^2(n) \longrightarrow Gl(n,\mathbb{R})$, $pr_3 : \hat{G}^2(n) \longrightarrow B^2(n)$, $pr_1 : G^2(n) \longrightarrow Gl(n,\mathbb{R})$ and $pr_3 : G^2(n) \longrightarrow S^2(n)$ the canonical projections.

From (3) we deduce that pr_1 is a Lie group homomorphism. However, pr_3 is not a Lie group homomorphism. Therefore, an arbitrary Lie subgroup $\hat{G}$ of $\hat{G}^2(n)$ (resp. $\check{G}$ of $G^2(n)$) may be written as follows: $\hat{G} = (G, \Sigma)$ (resp. $\check{G} = (G, \Sigma)$), where $G = pr_1(\hat{G})$ (resp. $G = pr_1(\check{G})$) is a Lie subgroup of $Gl(n,\mathbb{R})$, and $\Sigma = pr_3(\hat{G})$ (resp. $\Sigma = pr_3(\check{G})$) is a subset of $B^2(n)$ (resp. $S^2(n)$). Given an arbitrary element $A \in G$ we denote by Σ_A the subset of $B^2(n)$ (resp. $S^2(n)$) defined by

$$\Sigma_A = \{\alpha \in B^2(n) \mid (A,\alpha) \in \hat{G}\} \qquad (\text{resp. } \Sigma_A = \{\alpha \in S^2(n) | (A,\alpha) \in \check{G}\}).$$

It is easy to check that Σ_1 is an additive subgroup of $B^2(n)$ (resp. $S^2(n)$) and $(1, \Sigma_1)$ is a Lie subgroup of $\hat{G}^2(n)$ (resp. $G^2(n)$).

The following result is proved in a similar way than in Proposition 5.1.

Proposition 5.2. *For an arbitrary element $A \in G$ there exists a one-to-one correspondence between Σ_1 and Σ_A.*

6. Subgroups of $\bar{G}^2(n)$

Next, we shall give a classification of the Lie subgroups of $\bar{G}^2(n)$.

6.1. Toupin subgroups

Let G_1 and G_2 be two arbitrary Lie subgroups of $Gl(n,\mathbb{R})$ and $(1,1,\alpha) \in \bar{G}^2(n)$. A direct computation from (2) shows that $(G_1, G_2, 0)$ is a Lie subgroup of $\bar{G}^2(n)$. The conjugate subgroup of $(G_1, G_2, 0)$ by the element $(1, 1, \alpha)$ will be called a *Toupin subgroup*. We have

$$(1,1,\alpha)(G_1, G_2, 0)(1,1,\alpha)^{-1} = (G_1, G_2, \alpha(G_1, G_2) - G_1\alpha)$$

with the obvious notations.

6.2. Generalized Toupin subgroups

If $\bar{G} = (G_1, G_2, \Sigma)$ is a Toupin subgroup, we have $\Sigma_{(1,1)} = \{0\}$. Hence, we introduce the following definition. A subgroup $\bar{G} = (G_1, G_2, \Sigma)$ for which $\Sigma_{(1,1)} = \{0\}$ will be called a *generalized Toupin subgroup*. From Proposition 5.1 we deduce that $\Sigma_{(A,A')}$ is also a singleton.

Of course, a Toupin subgroup is a generalized Toupin subgroup. However, the converse is not true, as the next result proves.

Proposition 6.1. *Every one-parameter subgroup $\bar{G}$ of $\bar{G}^2(n)$ is a generalized Toupin subgroup with exception of the one-parameter subgroups of the form* $\exp t(0, 0, \alpha)$, $\alpha \neq 0$. *Furthermore, there exist one-parameter subgroups which are not Toupin subgroups.*

Proof. The one-parameter subgroups of $\bar{G}^2(n)$ are in one-to-one correspondence with the tangent vectors at $(1, 1, 0)$, or, in other words, with the Lie algebra $\bar{g}^2(n)$ of $\bar{G}^2(n)$. Let (A, A', α) be an element of $\bar{g}^2(n)$ such that A and A' do not simultaneously vanish. Then the one-parameter subgroup determined by (A, A', α) is

$$\exp t(A, A', \alpha) = (\exp tA, \exp tA', \phi(t, A, A', \alpha)),$$

where $\phi : \mathbb{R} \longrightarrow B^2(n)$. Then $\Sigma_{(\exp tA, \exp tA')} = \{\phi(t, A, A', \alpha)\}$, and thus $\{\exp t(A, A', \alpha)\}$ is a generalized Toupin subgroup.

Suppose now that $\exp t(A, A', \alpha)$ is a Toupin subgroup. Then it must be the conjugate subgroup of some $(G_1, G_2, 0)$ by an element of the form $(1, 1, \beta)$:

$$\exp t(A, A', \alpha) = (1, 1, \beta)(G_1, G_2, 0)(1, 1, \beta)^{-1},$$

or, equivalently,

$$(G_1, G_2, 0) = (1, 1, \beta)^{-1} \exp t(A, A', \alpha)(1, 1, \beta).$$

We obtain

$$\begin{aligned}(G_1, G_2, 0) = (&\exp tA, \exp tA', (\exp tA)\beta + \phi(t, A, A', \alpha) \\ &- \beta(\exp tA, \exp tA')).\end{aligned}$$

Hence,

$$(\exp tA)\beta + \phi(t, A, A', \alpha) - \beta(\exp tA, \exp tA') = 0. \tag{4}$$

If we differentiate (4) with respect to t at $t = 0$, we deduce

$$A\beta + \alpha - \beta(A, A') = 0. \tag{5}$$

Since A, A' and α are arbitrary, suppose that $A = A' = 1$ and $\alpha \neq 0$. Thus, from (5) we have $\alpha = 0$, which is a contradiction.

Moreover, there are one-parameter subgroups which are not generalized Toupin subgroups. For instance, we have $\exp t(0, 0, \alpha) = (1, 1, t\alpha)$. Then, if $\alpha \neq 0$ (in which case $\exp t(0, 0, 0)$ is the trivial subgroup $(1, 1, 0)$) we deduce that $(1, 1, t\alpha)$ is not a generalized Toupin subgroup. □

Remark 6.2. The Toupin subgroups are, therefore, rather the exception than the rule.

6.3. Conjugate subgroups of $(1, 1, \Sigma_{(1,1)})$

If $(1, 1, \Sigma_{(1,1)})$ is a subgroup of $\bar{G}^2(n)$, then $\Sigma_{(1,1)}$ is an additive subgroup of $B^2(n)$. The additive subgroups of an Euclidean space $\mathbb{R}^m$ have been completely classified by Morris [71].

As we know $(1, 1, \Sigma_{(1,1)})$ is closed if and only if $\Sigma_{(1,1)}$ is a closed additive subgroup of $B^2(n)$. Since we are primarily interested in closed subgroups of $\bar{G}^2(n)$ we only need to classify the closed subgroups of $B^2(n)$. Notice that $B^2(n)$ is isomorphic as a vector space to $\mathbb{R}^m$, where $m = n^3$.

Now, we recall the results of Morris [71]. If A is a subset of $\mathbb{R}^m$ we denote by $sp_{\mathbb{R}}(A)$ the span of A over $\mathbb{R}$, i.e., the subgroup

$$sp_{\mathbb{R}}(A) = \{t_1 a_1 + \cdots + t_s a_s, \mid t_1, \ldots, t_s \in \mathbb{R}, \ s \text{ is a positive integer}\}.$$

Notice that $sp_{\mathbb{R}}(A)$ is a vector subspace of $\mathbb{R}^m$. Then we define the *rank* of A to be the dimension of $sp_{\mathbb{R}}(A)$. Morris has proved the following result (see [71, Theorem 6, p. 33]):

Theorem 6.3. *Let Σ be a closed additive subgroup of $\mathbb{R}^m$. If the rank of Σ is r, then Σ is isomorphic to $\mathbb{R}^p \times \mathbb{Z}^{r-p}$, where $1 \leq p \leq r$. If Σ is discrete we have $\Sigma = \mathbb{Z}^r$.*

Hence the closed subgroups of $B^2(n) \cong \mathbb{R}^{n^3}$ are:
- discrete subgroups $\mathbb{Z}^r$,
- vector subspaces $\mathbb{R}^r$,
- or mixed subgroups $\mathbb{R}^p \times \mathbb{Z}^{r-p}$.

Consider the conjugate subgroups of $(1, 1, \Sigma_{(1,1)})$ by an arbitrary element $(A, A', \beta) \in \bar{G}^2(n)$. Then we obtain

$$(A, A', \beta)(1, 1, \Sigma_{(1,1)})(A, A', \beta)^{-1} = (1, 1, A\Sigma_{(1,1)}(A^{-1}, A'^{-1})).$$

Thus the element β is not relevant for conjugation of subgroups of the form $(1, 1, \Sigma_{(1,1)})$. Hence we shall only consider the conjugate subgroups obtained by conjugation of $(1, 1, \Sigma_{(1,1)})$ with two elements $A, A' \in Gl(n, \mathbb{R})$.

A similar classification can be given for the subgroups of $\hat{G}^2(n)$ and $G^2(n)$, but we omit here the details. Indeed, we have
- Toupin subgroups:
  $(G, \alpha(G, G) - G\alpha)$, where G is a subgroup of $Gl(n, \mathbb{R})$ and $\alpha \in B^2(n)$ (resp. $\alpha \in S^2(n)$);
- generalized Toupin subgroups:
  (G, Σ), where G is a subgroup of $Gl(n, \mathbb{R})$ and $\Sigma_1 = \{0\}$;
- subgroups of the form $(1, A\Sigma_1(A^{-1}, A^{-1}))$, where $A \in Gl(n, \mathbb{R})$ and Σ_1 is an additive subgroup of $B^2(n)$ (resp. $S^2(n)$).

7. Second-order frame bundles and linear connections

There exists a close relation between linear connections on a manifold M and invariant sections of the second-order non-holonomic, semi-holonomic and holonomic frame bundles over the linear frame bundle FM of M. In fact, roughly speaking, a non-holonomic frame of second order is a horizontal space of a linear connection. In this section we shall briefly recall the main results.

7.1. Sections of $\bar{F}^2M$

Let γ be an invariant global section of the second-order non-holonomic frame bundle $\bar{F}^2M$ over FM, i.e., $\gamma : FM \longrightarrow \bar{F}^2M$ such that

$$\bar{\pi}_1^2 \circ \gamma = id_{FM}, \qquad \gamma(zA) = \gamma(z)j(A) = \gamma(z)(A, A, 0),$$
$$\forall z \in FM, \quad \forall A \in Gl(n, \mathbb{R}).$$

In local coordinates we write $\gamma(x^i, x^i_j) = (x^i, x^i_j, \gamma^i_{,j}(x^a, x^a_b), \gamma^i_{j,k}(x^a, x^a_b))$.
The invariance of γ implies the following identities:

$$\gamma^i_{,j}(x^a, x^a_r A^r_b) = \gamma^i_{,s} A^s_j, \quad \gamma^i_{j,k}(x^a, x^a_r A^r_b) = \gamma^i_{r,s} A^r_j A^s_k. \tag{6}$$

The section γ defines a connection in the principal bundle FM as follows. Suppose that for an arbitrary point $z \in FM$ we have $\gamma(z) = j^1_{e_1, \Psi(e_1)} \Psi$, where $\Psi : F\mathbb{R}^n \longrightarrow FM$ is a local isomorphism, $\Psi(e_1) = z$. Hence, Ψ is locally written as $\Psi(r^i, r^i_j) = (\psi^i(r^a), \Psi^i_k(r^a, 1)r^k_j)$. Hence, we have a well-defined mapping $\Phi : \mathbb{R}^n \longrightarrow FM$, $\Phi(r) = \Psi(r, 1)$ for all $r \in \mathbb{R}^n$. In local coordinates we have $\Phi(r^i) = (\psi^i(r^a), \Psi^i_k(r^a, 1))$. Here (r^a, r^a_b) denotes the canonical coordinates in $F\mathbb{R}^n$. Now, we define a vector subspace at the point z by taking $H_z = \mathrm{d}\Phi(0)(T_0\mathbb{R}^n)$.

In local coordinates we obtain

$$\mathrm{d}\Phi(0)\left(\frac{\partial}{\partial r^k}\right) = \frac{\partial \psi^i}{\partial r^k}\frac{\partial}{\partial x^i} + \frac{\partial \Psi^i_j}{\partial r^k}\frac{\partial}{\partial x^i_j} = \gamma^i_{,k}\frac{\partial}{\partial x^i} + \gamma^i_{j,k}\frac{\partial}{\partial x^i_j}.$$

Thus, the vector subspace H_z is generated by the basis

$$\left\{ X_k = \gamma^i_{,k}\frac{\partial}{\partial x^i} + \gamma^i_{j,k}\frac{\partial}{\partial x^i_j} \right\}.$$

These vector subspaces are horizontal (i.e. they are complementary to the vertical subspace at this point). Therefore, we obtain a smooth distribution H on FM and hence a connection Γ in the principal bundle $\pi_M : FM \longrightarrow M$. Furthermore, this connection is linear because the horizontal distribution is invariant by the action of $Gl(n, \mathbb{R})$.

Next, we shall compute the Christoffel components of Γ. First, notice that the local vector fields

$$Y_r = (\gamma^{-1})^k_{,r} X_k = \frac{\partial}{\partial x^r} + (\gamma^{-1})^k_{,r}\gamma^i_{j,k}\frac{\partial}{\partial x^i_j}$$

form a local basis of H. Taking into account that the horizontal lift of $\partial/\partial x^r$ to FM with respect to Γ is

$$\left(\frac{\partial}{\partial x^r}\right)^H = \frac{\partial}{\partial x^r} - \Gamma^i_{ra} x^a_j \frac{\partial}{\partial x^i_j},$$

we deduce that

$$\Gamma^i_{rt} = -\gamma^i_{j,k}(x^{-1})^j_t(\gamma^{-1})^k_{,r},$$

where $(x^{-1})^i_j$ denotes the inverse matrix of (x^i_j). Observe that, in fact, Γ^i_{rt} does not depend on the choice of x^i_j.

7.2. Sections of $\hat{F}^2M$

Now, let us suppose that γ takes values into $\hat{F}^2M$, or, in other words, γ is a global invariant section of the second-order semi-holonomic frame bundle $\hat{F}^2M$ over FM, i.e., $\hat{\pi}^2_1 \circ \gamma = id$ and $\gamma(zA) = \gamma(u)j(A)$ for all $z \in FM$, $A \in Gl(n, \mathbb{R})$.

The section γ is in particular a global invariant section of $\bar{F}^2M$ and hence it induces a connection Γ in FM.

If we write $\gamma(x^i, x^i_j) = (x^i, x^i_j, \gamma^i_j(x^a, x^a_b) = x^i_j, \gamma^i_{j,k}(x^a, x^a_b))$, then the vector subspace H_z, at a linear frame $z \in FM$, is generated by the basis

$$\left\{ X_k = x^i_k \frac{\partial}{\partial x^i} + \gamma^i_{j,k} \frac{\partial}{\partial x^i_j} \right\}.$$

Proceeding as above we obtain the Christoffel components of Γ:

$$\Gamma^i_{rt} = -\gamma^i_{j,k}(x^{-1})^j_t(x^{-1})^k_r,$$

where $(x^{-1})^i_j$ denotes the inverse matrix of (x^i_j).

Conversely, if Γ is a linear connection on M, then we can construct a global invariant section $\gamma : FM \longrightarrow \hat{F}^2M$ as follows.

Let $z \in FM$ be an arbitrary linear frame at a point $x \in M$. Denote by H_z the horizontal subspace defined by Γ at z. If we consider local coordinates (x^i) on M then a local basis $\{Y_k\}$ of H_z is given by

$$Y_k = \frac{\partial}{\partial x^k} - \Gamma^i_{k,a} x^a_j \frac{\partial}{\partial x^i_j},$$

where $\Gamma^i_{k,a}$ are the Christoffel components of Γ in the given coordinate system. We now change this basis to the following:

$$\left\{ X_r = x^k_r Y_k = x^k_r \frac{\partial}{\partial x^k} - \Gamma^i_{k,a} x^a_j x^k_r \frac{\partial}{\partial x^i_j} \right\}.$$

This new basis of H_z may be completed to a basis of the whole tangent space $T_z(FM)$ by taking the standard basis of the vertical subspace at z, namely

$$\left\{ X^r_s = x^s_a \frac{\partial}{\partial x^r_a} \right\}.$$

In fact, $\{X^r_s\}$ are the fundamental vector fields induced by the canonical basis of the Lie algebra $gl(n, \mathbb{R})$ of $Gl(n, \mathbb{R})$.

Thus, we have obtained a linear frame $\tilde{z}$ of FM at the point z which may be locally represented in induced coordinates as follows:

$$\tilde{z} = (x^i, x^i_j; x^i_j, x^i_{,jk} = 0, \gamma^i_{j,k} = -\Gamma^i_{rs} x^r_k x^s_j, x^i_{j,kl} = x^i_k \delta_{jl}).$$

Therefore, $\tilde{z}$ is a second-order semi-holonomic frame at z and we obtain a global invariant section $\gamma : FM \longrightarrow \hat{F}^2 M$ locally defined by $\gamma(x^i, x^i_j) = (x^i, x^i_j, \gamma^i_{j,k}(x^a, x^a_b))$.

Remark 7.1. We can obtain γ from Γ in a different way as follows. Denote by $\alpha : T_x M \longrightarrow H_z$ the inverse map induced by the connection and suppose that $z = j^1_0 \phi$. α can be realized by a local section $\sigma : M \longrightarrow FM$, i.e., $\sigma(x) = z$ and $d\sigma(x) = \alpha$.

Hence we consider the local bundle isomorphism $\Phi(r^a, r^a_b) = (\phi^i(r^a), \sigma^i_k(\phi(r^a)) r^k_j)$. A direct computation shows that $j^1_{e_1} \Phi = \tilde{z}$.

7.3. Sections of $F^2 M$

Now, let us suppose that γ takes values into $F^2 M$, i.e., γ is a global invariant section of the second-order frame bundle $F^2 M$ over FM. From the above sections, we deduce that γ induces a linear connection Γ on M. In local coordinates we have $\gamma(x^i, x^i_j) = (x^i, x^i_j, \gamma^i_j(x^a, x^a_b) = x^i_j, \gamma^i_{jk}(x^a, x^a_b))$, with $\gamma^i_{jk} = \gamma^i_{kj}$.

The Christoffel components of the linear connection Γ are:

$$\Gamma^i_{rt} = -\gamma^i_{jk} (x^{-1})^j_t (x^{-1})^k_r,$$

and thus Γ is symmetric.

Conversely, if Γ is a symmetric linear connection on M, then the global invariant section $\gamma : FM \longrightarrow \hat{F}^2 M$ takes values into $F^2 M$.

Summarizing the results of the last three subsections, we have the following (see Libermann [61], Yuen [86], de León and Ortacgil [21]):

Theorem 7.2.

(i) *An invariant section $\gamma : FM \longrightarrow \bar{F}^2 M$ of $\bar{\pi}^2_1$ induces a linear connection on M.*

(ii) *There exists a one-to-one correspondence between linear connections on M and invariant sections $\gamma : FM \longrightarrow \hat{F}^2 M$.*

(iii) *There exists a one-to-one correspondence between symmetric linear connections on M and invariant sections $\gamma : FM \longrightarrow F^2 M$.*

Remark 7.3. If we begin with an invariant section $\gamma : FM \longrightarrow \bar{F}^2 M$ then we obtain a linear connection Γ with Christoffel components

$$\Gamma^i_{rt} = -\gamma^i_{j,k} (x^{-1})^j_t (\gamma^{-1})^k_{,r},$$

where $\gamma(x^i, x^i_j) = (x^i, x^i_j, \gamma^i_{,j}(x^a, x^a_b), \gamma^i_{j,k}(x^a, x^a_b))$.

From Theorem 7.2, we deduce that Γ induces an invariant section $\sigma : FM \longrightarrow \hat{F}^2M$ locally expressed by $\sigma(x^i, x^i_j) = (x^i, x^i_j, \sigma^i_{j,k}(x^a, x^a_b))$.

A direct computation shows that $\sigma(x^i, x^i_j) = \gamma(x^i, x^i_j)(1, (x^{-1})^i_a\gamma^a_{,j}, 0)$, which can be written in an intrinsic way taking into account that $(x^{-1})^i_a\gamma^a_{,j} = (\bar{\pi}^2_1(\gamma(z)))^{-1}\tilde{\pi}^2_1(\gamma(z))$, for all $z \in FM$. Therefore we have

$$\sigma(z) = \gamma(z)(1, \tau(z), 0), \quad \forall z \in FM,$$

where $\tau : FM \longrightarrow Gl(n, \mathbb{R})$ is defined by $\tau(z) = (\bar{\pi}^2_1(\gamma(z)))^{-1}\tilde{\pi}^2_1(\gamma(z))$.

7.4. Non-holonomic prolongations of linear parallelisms

We shall describe a method to prolongate a pair of linear parallelisms in order to obtain a non-holonomic parallelism of second order.

Let $\bar{P}$ be a non-holonomic parallelism of second order on a manifold M. Then $\bar{P}$ induces two ordinary parallelisms P and Q on M by projecting $\bar{P}$ via the two canonical projections $\bar{\pi}^2_1 : \bar{F}^2M \longrightarrow FM$ and $\tilde{\pi}^2_1 : \bar{F}^2M \longrightarrow FM$.

If $\bar{P}(x^i) = (x^i, P^i_j, Q^i_{,j}, R^i_{j,k})$, then we obtain

$$P(x^i) = (x^i, P^i_j), \qquad Q(x^i) = (x^i, Q^i_{,j}).$$

Conversely, let P, Q be two linear parallelisms on a manifold M. Hence, P (resp. Q) defines a set of n linearly independent vector fields $\{P_1, \dots, P_n\}$ (resp. $\{Q_1, \dots, Q_n\}$).

We define a horizontal subspace $H_{P(x)}$ at the point $P(x)$ by translating the basis $\{Q_a(x)\}$ at x into a set of linearly independent tangent vectors $\{\mathrm{d}P(x)(Q_a(x))\}$ at $P(x)$.

In local coordinates we obtain

$$\mathrm{d}P(x)\left(Q^i_a\frac{\partial}{\partial x^i}\right) = Q^i_a\frac{\partial}{\partial x^i} + Q^i_a\frac{\partial P^r_s}{\partial x^i}\frac{\partial}{\partial x^r_s},$$

where

$$P_a = P^i_a\frac{\partial}{\partial x^i}, \qquad Q_a = Q^i_a\frac{\partial}{\partial x^i}.$$

By completing this set of linearly independent tangent vectors to a basis of $T_{P(x)}(FM)$ we obtain a second-order non-holonomic frame at x. We have so obtained a non-holonomic parallelism of second order, which will be denoted by $P^1(Q)$.

Definition 7.4. A non-holonomic parallelism of second-order $\bar{P}$ is said to be a *prolongation* if $\bar{P} = P^1(Q)$, where P and Q are the induced ordinary parallelisms.

The local expression of $P^1(Q)$ becomes

$$P^1(Q)(x^i) = \left(x^i, P^i_j, Q^i_j, Q^u_k\frac{\partial P^i_j}{\partial x^u}\right).$$

Hence, $\bar{P}$ is a prolongation if and only if we have

$$R^i_{j,k} = Q^u_k \frac{\partial P^i_j}{\partial x^u}.$$

Notice that, if Q is integrable, then there exists local coordinates (x^i) on M such that $Q^i_j = \delta^i_j$ and $R^i_{j,k} = (\partial P^i_j)/(\partial x^k)$, where $\bar{P}(x^i) = (x^i, P^i_j, Q^i_j, R^i_{j,k})$. In such a case, $\bar{P}$ is said to be an *integrable prolongation*.

There exists a geometric way in order to decide if a second-order non-holonomic parallelism $\bar{P}$ is a prolongation or not. In fact, the induced parallelisms P and Q define two linear connections, respectively, denoted by Γ_1 and Γ_2. We briefly recall their construction.

If $P = \{P_1, \ldots, P_n\}$ and $Q = \{Q_1, \ldots, Q_n\}$, then Γ_1 is defined by its covariant derivative:

$$(\nabla_1)_{P_a} P_b = 0,$$

and, in a similar way, we define Γ_2 by imposing

$$(\nabla_2)_{Q_a} Q_b = 0.$$

Here ∇_1 and ∇_2 are the covariant derivatives defined by Γ_1 and Γ_2, respectively.

In other words, we transport the tangent space $T_x M$ by means of $\mathrm{d}P(x)$ and obtain a horizontal subspace at the point $P(x)$ for every $x \in M$ and, then, we extend the horizontal spaces so obtained by the action of the Lie group $Gl(n, \mathbb{R})$. The same is true for the parallelism Q.

The Christoffel components of Γ_1 and Γ_2 are respectively:

$$(\Gamma_1)^i_{jk} = -(P^{-1})^a_k \frac{\partial P^i_a}{\partial x^j}, \qquad (\Gamma_2)^i_{jk} = -(Q^{-1})^a_{.k} \frac{\partial Q^i_{.a}}{\partial x^j}.$$

The two connections Γ_1 and Γ_2 are flat, but in general they have non-zero torsion. As we know, the integrability of the parallelisms P and Q are equivalent to the vanishing of their torsion tensors.

Remark 7.5. Notice that the horizontal subspaces at the points $P(x)$ defined from the non-holonomic parallelism $P^1(Q)$ are just the ones corresponding to the linear connection Γ_1.

Moreover, the non-holonomic parallelism $\bar{P}$ induces an ordinary parallelism $\tilde{P}$ on FM as follows. We define

$$\tilde{P}(P(x)) = \bar{P}(x), \qquad \tilde{P}(P(x)A) = \bar{P}(x)(j(A)) = \bar{P}(x)(A, A, 0).$$

Notice that $\tilde{P}$ takes values into $\bar{F}^2 M$ and, thus, it is in fact an invariant global section of $\bar{\pi}^2_1 : \bar{F}^2 M \longrightarrow FM$ which is locally expressed by

$$\tilde{P}(x^i, x^i_j) = (x^i, x^i_j, Q^i_{.k}(P^{-1})^k_t x^t_j, R^i_{r.s}(P^{-1})^r_u x^u_j (P^{-1})^s_v x^v_k).$$

According to Section 7.1, $\tilde{P}$ induces a linear connection Λ on M whose Christoffel components are

$$\Lambda^i_{jk} = -R^i_{r,s}(P^{-1})^r_k(Q^{-1})^s_{,j}.$$

Thus, we have obtained from $\bar{P}$ three linear connections Γ_1, Γ_2 and Λ. The following result follows by a direct computation in local coordinates.

Proposition 7.6. *A second-order non-holonomic parallelism $\bar{P}$ is a prolongation if and only if the two connections Γ_1 and Λ coincide.*

Corollary 7.7. *A second-order non-holonomic parallelism $\bar{P}$ is an integrable prolongation if and only if Γ_2 has no torsion and the two connections Γ_1 and Λ coincide.*

The preceding corollary may be equivalently stated as follows. Let T_2 be the tensor torsion of Γ_2 and $D = \Gamma_1 - \Lambda$ the difference tensor. Then we have the following:

Corollary 7.8.
(1) *A second-order non-holonomic parallelism $\bar{P}$ is a prolongation if and only if D identically vanishes.*
(2) *A second-order non-holonomic parallelism $\bar{P}$ is an integrable prolongation if and only if T_2 and D simultaneously vanish.*

Corollary 7.9. *An integrable second-order non-holonomic parallelism is an integrable prolongation. Furthermore, a second-order semi-holonomic parallelism is an integrable prolongation if and only if it is integrable.*

Remark 7.10. Notice that the parallelism $\tilde{P}$ on FM defines a set $\{\tilde{P}_a, \tilde{P}^a_b\}$ of linearly independent vector fields on FM:

$$\tilde{P}_a = Q^i_{,a}\frac{\partial}{\partial x^i} + R^i_{j,k}\frac{\partial}{\partial x^i_j}, \qquad \tilde{P}^a_b = P^i_a\frac{\partial}{\partial x^i_b}.$$

Now, suppose that $\bar{\omega}_{\bar{G}}(M)$ is a non-holonomic $\bar{G}$-structure of second order on M.

Definition 7.11. We say that $\bar{\omega}_{\bar{G}}(M)$ is an *integrable prolongation* if there exists an adapted local section which is a non-holonomic integrable prolongation.

Proposition 7.12. *If $\bar{\omega}_{\bar{G}}(M)$ is integrable, then it is an integrable prolongation. Conversely, if $\hat{\omega}_{\hat{G}}$ is an integrable prolongation second-order semi-holonomic $\hat{G}$-structure, then it is integrable.*

It directly follows that if $\bar{\omega}_{\bar{G}}(M)$ is an integrable prolongation, then the projected G-structure $\omega_G(M)$ is integrable.

8. Jet of mappings and jet manifolds

In this section, we shall give a brief review on jet manifolds (see [5,50], for instance).

Let M and N be C^∞ manifolds of dimension m and n, respectively. Two C^∞ mappings $f, g : M \longrightarrow N$ are said to be *k-equivalent* at a point $x \in M$ if their kth Taylor expansions at x agree. In this case, we say that f and g define the same *k-jet* $j_x f$ (or $j_{x,f(x)} f$).

Consider the set $J^k(M, N)$ of all k-jets $j_x f$ of all mappings from M to N. If we choose local coordinates (x^i) on M and (y^α) on N, we obtain local coordinates $(x^i, y^\alpha, y^\alpha_{i_1 \cdots i_r})$ for $J^k(M, N)$, where

$$y^\alpha_{i_1 \cdots i_r} = \frac{\partial^{i_1 + \cdots + i_r} f^\alpha}{\partial x^{i_1} \cdots \partial x^{i_r}}$$

for any r-tuple $(i_1, \ldots, i_r)$ such that $i_1 + \cdots + i_r \leq k$. Thus, $J^k(M, N)$ becomes a C^∞ manifold.

Notice that $J^k(M, N)$ has several fibred structures. In fact, if $r \leq k$, there exists a canonical projection $\pi^k_r : J^k(M, N) \longrightarrow J^r(M, N)$ defined by $\pi^k_r(j^k_x f) = j^r_x f$. Also, there are canonical projections $\alpha : J^k(M, N) \longrightarrow M$ and $\beta : J^k(M, N) \longrightarrow N$, given by $\alpha(j^k_x f) = x$, $\beta(j^k_x f) = f(x)$; α and β are called the *source* and *target* mappings, respectively.

Let $f : M \longrightarrow N$ be a C^∞ mapping. We define the *k-jet prolongation* of f as the mapping $j^k f : M \longrightarrow J^k(M, N)$ given by $j^k f(x) = j^k_x f$ for any $x \in M$.

9. Lie groupoids

Let us recall the definition of groupoid (we refer the reader to [64] for a good reference on groupoids; see also [23–26,28,29,62,63]).

Let $\mathcal{B}$ a set. A *groupoid* over $\mathcal{B}$ is a set Ω provided with two maps $\alpha : \Omega \longrightarrow \mathcal{B}$ and $\beta : \Omega \longrightarrow \mathcal{B}$ and a law of composition satisfying the following conditions:

(i) For $Z, Z' \in \Omega$, the product $Z \cdot Z'$ is defined if and only if $\alpha(Z) = \beta(Z')$, and then $\beta(Z \cdot Z') = \beta(Z), \alpha(Z \cdot Z') = \alpha(Z')$.

(ii) The triple product $Z \cdot (Z' \cdot Z'')$ is defined if and only if $(Z \cdot Z') \cdot Z''$ is defined and, when one of them is defined, the associative law $Z \cdot (Z' \cdot Z'') = (Z \cdot Z') \cdot Z''$ holds.

(iii) For each $X \in \mathcal{B}$, there exists an element $1_X \in \Omega$ such that
  (a) $\alpha(1_X) = \beta(1_X) = X$,
  (b) if $Z \cdot 1_X$ is defined, then $Z \cdot 1_X = Z$,
  (c) if $1_X \cdot Z$ is defined, then $1_X \cdot Z = Z$.

(iv) For each $Z \in \Omega$ there exists $Z^{-1} \in \Omega$ such that $Z^{-1} \cdot Z = 1_X, Z \cdot Z^{-1} = 1_Y$, where $X = \alpha(Z), Y = \beta(Z)$.

From the definition it follows that for every element $X \in \mathcal{B}$ there exists a unique *unity* 1_X, and every element $Z \in \Omega$ has a unique *inverse* Z^{-1}. The set $\mathcal{B}$ is called the subset of *unities* of Ω.

A subset Ω' of a groupoid Ω is called a *subgroupoid* if Ω' is a groupoid with respect to the law of composition of Ω.

Next, we shall introduce differentiability. A groupoid Ω over $\mathcal{B}$ is called a *differentiable groupoid* if:

(i) Ω and $\mathcal{B}$ are differentiable manifolds.
(ii) The maps $\alpha : \Omega \longrightarrow \mathcal{B}$ and $\beta : \Omega \longrightarrow \mathcal{B}$ are submersions (and hence they are differentiable).
(iii) The map $Z \longrightarrow Z^{-1}$ is differentiable (and hence a diffeomorphism).
(iv) For any differentiable manifold N and for two differentiable maps $f, g : N \longrightarrow \Omega$ such that $\alpha \circ f = \beta \circ h$, the map $f \cdot h : N \longrightarrow \Omega$ defined by $(f \cdot h)(u) = f(u) \cdot h(u)$ is differentiable. Hence, the product $(Z, Z') \rightsquigarrow Z \cdot Z'$, which is defined on the submanifold $A = \{(Z, Z') | \beta(Z) = \alpha(Z')\} \subset \Omega$, is differentiable.

Suppose that Ω is a differentiable groupoid. Ω is called a *Lie groupoid* if the map $(\alpha, \beta) : \Omega \longrightarrow \mathcal{B} \times \mathcal{B}$, $(\alpha, \beta)(Z) = (\alpha(Z), \beta(Z))$ is a submersion. Notice that if (α, β) is a submersion, then α and β are also submersions. If (α, β) is also surjective, then Ω is called a *transitive Lie groupoid*. If Ω' is a submanifold of Ω such that Ω' is a subgroupoid of Ω and a Lie groupoid over $\mathcal{B}$, then Ω' is called a *Lie subgroupoid* of Ω.

We now give some examples of Lie groupoids.

Example 9.1. Let M be an n-dimensional manifold and denote by $\Pi^1(M, M)$ the manifold of the 1-jets $j^1_{x,y}\phi$ of local diffeomorphisms ϕ from M to M. A direct computation shows that $\Pi^1(M, M)$ is a Lie groupoid over M with the source and target maps, respectively, defined by $\alpha(j^1_{x,y}\phi) = x$ and $\beta(j^1_{x,y}\phi) = y$.

Example 9.2. Let P be a principal bundle over a manifold M with structure group G and projection $\pi : P \longrightarrow M$. We denote by $J^1(P)$ the manifold of 1-jets $j^1_{u,\Phi(u)}\Phi$ of local automorphisms Φ of P such that $\Phi(va) = \Phi(v)a\ \forall v \in P, \forall a \in G$. Notice that $J^1(P) \subset \Pi^1(P, P)$. We define an equivalence relation on $J^1(P)$ as follows: $j^1_{u,\Phi(u)}\Phi \sim j^1_{ua,\Phi(u)a}\Phi$. Denote by $\tilde{J}^1(P)$ the quotient space $J^1(P)/G$. If we define

$$\tilde{\alpha}([j^1_{u,\Phi(u)}\Phi]) = \pi(u), \qquad \tilde{\beta}([j^1_{u,\Phi(u)}\Phi]) = \pi(\Phi(u)),$$

we can easily check that $\tilde{J}^1(P)$ is a Lie groupoid over M with source and target maps $\tilde{\alpha}, \tilde{\beta} : \tilde{J}^1(P) \longrightarrow M$. Sometimes we will denote by $j^1_{x,\phi(x)}\Phi$ the equivalence class of $j^1_{u,\Phi(u)}\Phi$, where $x = \pi(u)$. With some abuse of notation $j^1_{x,\phi(x)}\Phi$ will be called the 1-jet of Φ at x.

Part II. Cosserat media

10. Configurations and all that

10.1. Configurations of Cosserat media

A *body* $\mathcal{B}$ is a three-dimensional differentiable manifold which can be covered with just one chart. An embedding $\phi : \mathcal{B} \longrightarrow \mathbb{R}^3$ is called a *configuration* of $\mathcal{B}$ and its 1-jet $j^1_{X,\phi(X)}\phi$

at $X \in \mathcal{B}$ is called an *infinitesimal configuration* at X. We usually identify the body with any one of its configurations, say $\phi_0 : \mathcal{B} \longrightarrow \mathbb{R}^3$, called a *reference configuration*. Given any arbitrary configuration $\phi : \mathcal{B} \longrightarrow \mathbb{R}^3$, the change of configurations $\kappa = \phi \circ \phi_0^{-1}$ is called a *deformation*, and its 1-jet $j^1_{\phi_0(X),\phi(X)}\kappa$ is called an *infinitesimal deformation* at $\phi_0(X)$.

For elastic bodies, the material is completely characterized by one function W which depends, at each point of $\mathcal{B}$, on the gradient of the deformation evaluated at that point, namely,

$$W = W(j^1_{X,\kappa(X)}\kappa) . \tag{7}$$

The picture describing a Cosserat medium is more complicated. In fact, a *Cosserat medium* is the linear frame bundle $F\mathcal{B}$ of a body $\mathcal{B}$. $\mathcal{B}$ is usually called the *macromedium* or *underlying body*. With some abuse of notation, we shall call $\mathcal{B}$ the Cosserat continuum.

A *configuration* of a Cosserat medium $\mathcal{B}$ is an embedding $\Psi : F\mathcal{B} \longrightarrow F\mathbb{R}^3$ of principal bundles such that the induced Lie group monomorphism $\tilde{\psi} : Gl(3, \mathbb{R}) \longrightarrow Gl(3, \mathbb{R})$ is the identity map. Hence, $\Psi : F\mathcal{B} \longrightarrow F\mathbb{R}^3$ is a morphism of principal bundles such that $\Psi(\tilde{X}a) = \Psi(\tilde{X})a$ for all $\tilde{X} \in F\mathcal{B}, a \in Gl(3, \mathbb{R})$. Also, Ψ induces a differentiable mapping $\psi : \mathcal{B} \longrightarrow \mathbb{R}^3$ in such a way that Ψ covers ψ. The mapping ψ is an embedding of $\mathcal{B}$ into $\mathbb{R}^3$. In particular, $\psi : \mathcal{B} \longrightarrow \mathbb{R}^3$ is a configuration of the underlying body $\mathcal{B}$.

Remark 10.1. The condition $\Psi(\tilde{X}a) = \Psi(\tilde{X})a$ means that Ψ transports the tangent space $T_X\mathcal{B}$ of $\mathcal{B}$ at $X = \pi_{\mathcal{B}}(\tilde{X})$ onto the tangent space $T_{\psi(X)}\mathbb{R}^3 \cong \mathbb{R}^3$ of $\mathbb{R}^3$ at $\psi(X)$. In fact, if $\tilde{X}$ is a frame at X, i.e., a basis of $T_X\mathcal{B}$, then $\Psi(\tilde{X})$ is a basis of $\mathbb{R}^3$. The above condition implies that this linear mapping does not depend on the choice of the linear frame $\tilde{X}$.

On the other hand, we have another linear isomorphism $\mathrm{d}\psi(X) : T_X\mathcal{B} \longrightarrow T_{\psi(X)}\mathbb{R}^3 \cong \mathbb{R}^3$.

Notice that the sub-bundle $\Psi(F\mathcal{B})$ of $F\mathbb{R}^3$ is just the frame bundle of $\psi(\mathcal{B})$, i.e., $\Psi(F\mathcal{B}) = F(\psi(\mathcal{B}))$.

Since we are dealing with equivariant embeddings, we can consider equivalence classes of the 1-jets $j^1_{\tilde{X},\Psi(\tilde{X})}\Psi$ according to Example 9.2. So, the 1-jet $j^1_{X,\psi(X)}\Psi$ is called an *infinitesimal configuration* at X. We usually identify the Cosserat medium with any one of its configurations, say $\Psi_0 : F\mathcal{B} \longrightarrow F\mathbb{R}^3$, and we denote by ψ_0 the induced mapping $\psi_0 : \mathcal{B} \longrightarrow \mathbb{R}^3$. Notice that $\Psi_0(F\mathcal{B}) = F(\psi_0(\mathcal{B}))$. $\Psi_0 : F\mathcal{B} \longrightarrow F\mathbb{R}^3$ is called a *reference configuration*. Given any arbitrary configuration, Ψ, the change of configuration $\tilde{\kappa} = \Psi \circ \Psi_0^{-1}$ is called a *deformation*, and its 1-jet $j^1_{\psi_0(X),\psi(X)}\tilde{\kappa}$ is called an *infinitesimal deformation* at $\psi_0(X)$. Notice that a deformation is a principal bundle isomorphism. We have of course a change of configuration of the underlying body $\mathcal{B}$, namely $\kappa = \psi \circ \psi_0^{-1}$, with the obvious notations for the induced mappings.

From now on we make the following identifications: $\mathcal{B} \cong \psi_0(\mathcal{B})$ and $F\mathcal{B} \cong \Psi_0(F\mathcal{B}) = F(\psi_0(\mathcal{B}))$.

Remark 10.2. A more general Cosserat media may be considered. In fact, we may consider deformations $\tilde{\kappa}$ such that $\tilde{\kappa}(\tilde{X}a) = \tilde{\kappa}(\tilde{X})\varphi(a)$, where $\varphi : Gl(3, \mathbb{R}) \longrightarrow Gl(3, \mathbb{R})$ is a Lie group isomorphism.

Our assumption is that the material is completely characterized by one function W which depends, at each point of $\mathcal{B}$, on the 1-jet of the deformation evaluated at the point X, namely,

$$W = W(j^1_{X,\kappa(X)}\tilde{\kappa}). \tag{8}$$

Eq. (8) is called the *constitutive law* of the Cosserat continuum.

The function W measures, for instance, the stored energy per unit mass.

Remark 10.3. A Hamiltonian description for elastic simple bodies can be found in [6,45,65,66]. The corresponding description for media with microstructure was recently studied in [7].

10.2. Uniform Cosserat media. Material symmetries

Suppose that an infinitesimal neighbourhood of the material around point Y can be grafted so perfectly into a neighbourhood of X, that the graft cannot be detected by any mechanical experiment. If this condition is satisfied with every point X of $\mathcal{B}$, the Cosserat medium is said to be *uniform*. This physical property can be expressed in a geometrical way as follows.

Definition 10.4. A Cosserat continuum $\mathcal{B}$ is said to be *uniform* if for two arbitrary points X and Y in $\mathcal{B}$ there exists a local principal bundle isomorphism Ψ from FU onto FV, where U is an open neighbourhood of X and V is an open neighbourhood of Y such that $\Psi(\tilde{Z}a) = \Psi(\tilde{Z})a$, $\tilde{Z} \in FU$, $a \in Gl(3, \mathbb{R})$, the induced local diffeomorphism $\psi : V \longrightarrow U$ maps X into Y, and

$$W(j^1_{Y,\kappa(Y)}\tilde{\kappa}) = W(j^1_{Y,\kappa(Y)}\tilde{\kappa} \cdot j^1_{X,Y}\Psi) \tag{9}$$

for all infinitesimal deformations $j^1_{Y,\kappa(Y)}\tilde{\kappa}$.

Denote by $\bar{G}(X,Y)$ the collection of all 1-jets $j^1_{X,\psi(X)}\Psi$ satisfying Eq. (9). So, $\bar{\Omega}(\mathcal{B})$ is a subset of the Lie groupoid $\tilde{J}^1(F\mathcal{B})$, and if the Cosserat continuum $\mathcal{B}$ is uniform then $\bar{\Omega}(\mathcal{B})$ is a transitive subgroupoid of $\tilde{J}^1(F\mathcal{B})$. Our assumption is that $\bar{\Omega}(\mathcal{B})$ is in fact a Lie subgroupoid, and this condition is the mathematical translation of the smooth uniformity.

We denote by $\bar{\alpha} : \bar{\Omega}(\mathcal{B}) \longrightarrow \mathcal{B}$ and $\bar{\beta} : \bar{\Omega}(\mathcal{B}) \longrightarrow \mathcal{B}$ the source and target mappings, respectively, which are in fact the restrictions of $\tilde{\alpha}$ and $\tilde{\beta}$. That is, we have $\bar{\alpha}(j^1_{X,\psi(X)}\Psi) = X$ and $\bar{\beta}(j^1_{X,\psi(X)}\Psi) = \psi(X))$.

Definition 10.5. Given a material point $X \in \mathcal{B}$ a *material symmetry* at X is a 1-jet $j^1_{X,\psi(X)}\Psi$, where Ψ is a local automorphism of $F\mathcal{B}$ at X such that $\Psi(\tilde{Y}a) = \Psi(\tilde{Y})a$ $\forall\tilde{Y} \in F\mathcal{B}, \forall a \in Gl(3, \mathbb{R})$, X is fixed by the induced local diffeomorphism ψ, and

$$W(j^1_{X,\kappa(X)}\tilde{\kappa}) = W(j^1_{X,\kappa(X)}\tilde{\kappa} \cdot j^1_{X,X}\Psi) \tag{10}$$

for all $j^1_{X,\kappa(X)}\tilde{\kappa}$.

We denote by $\bar{G}(X)$ the set of all material symmetries. It is easy to check that $\bar{G}(X)$ is a group with the composition of jets which is called the *isotropy group* or *group of material symmetries* at X.

Now, fix a point X_0 in $\mathcal{B}$ and put $\bar{\Omega}_{X_0}(\mathcal{B}) = \bar{\alpha}^{-1}(X_0)$. Then we deduce the following.

Proposition 10.6.
(i) $\bar{G}(X_0)$ *is a Lie group.*
(ii) $\bar{\Omega}_{X_0}(\mathcal{B})$ *is a principal bundle over* $\mathcal{B}$ *with structure group* $\bar{G}(X_0)$ *and projection* $\bar{\beta}$.

Proof. Since

$$\bar{\Omega}_{X_0}(\mathcal{B}) = \bar{\alpha}^{-1}(X_0),$$

we deduce that $\bar{\Omega}_{X_0}(\mathcal{B})$ is closed and in fact a closed submanifold of $\bar{\Omega}(\mathcal{B})$, since $\bar{\alpha}$ is a surjective submersion. Furthermore, we have

$$\bar{G}(X_0) = (\bar{\alpha}, \bar{\beta})^{-1}(X_0, X_0),$$

and then $\bar{G}(X_0)$ is a closed submanifold, since $\bar{\alpha} \times \bar{\beta}$ is a surjective submersion. Hence, $\bar{G}(X_0)$ is a Lie group from the Cartan theorem.

There exists an action of $\bar{G}(X_0)$ on $\bar{\Omega}_{X_0}(\mathcal{B})$ on the right which is given by composition of jets. Since $(\bar{\alpha}, \bar{\beta}) : \bar{\Omega}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ is a surjective submersion there exists an open covering $\{U_a\}$ of $\mathcal{B}$ and local sections of $(\bar{\alpha}, \bar{\beta})$, $\sigma_{a,b} : U_a \times U_b \longrightarrow \bar{\Omega}(\mathcal{B})$.

Suppose that $X_0 \in U_{a_0}$ and define $\sigma_a : U_a \longrightarrow \bar{\Omega}_{X_0}(\mathcal{B})$ by $\sigma_a(X) = \sigma_{a_0,a}(X_0, X)$. We obtain diffeomorphisms $\Lambda_a : U_a \times \bar{G}(X_0) \longrightarrow (\bar{\beta})^{-1}(U_a)$ defined by $\Lambda_a(X, Z) = \sigma_a(X) \cdot Z$. A direct computation shows that the family $\{U_a, \Lambda_a\}$ defines a principal bundle structure on $\bar{\Omega}_{X_0}(\mathcal{B})$ with structure group $\bar{G}(X_0)$ and projection $\bar{\beta}$, for which $\{\Lambda_a\}$ are local trivializations. The local sections $\{\sigma_a\}$ are adapted for the $\bar{G}(X_0)$-bundle structure. □

We have proved Proposition 10.6 by using a slight modification of the standard proof in the case of simple bodies (see [18,19] and the book of Fujimoto [49]).

A local section $\sigma : U \subset \mathcal{B} \times \mathcal{B} \longrightarrow \bar{\Omega}(\mathcal{B})$ of $(\bar{\alpha}, \bar{\beta})$, where U is an open subset of $\mathcal{B} \times \mathcal{B}$, will be called a *local uniformity*. In such a case we say that $\mathcal{B}$ enjoys *locally smooth uniformity*. A global section σ will be called a *global uniformity*, and, in that case, we say that $\mathcal{B}$ enjoys *smooth global uniformity*.

The assumption of the Lie groupoid character of $\bar{\Omega}(\mathcal{B})$ is, in fact, the mathematical translation of the smooth uniformity.

Next, we consider the set $\Omega(\mathcal{B})$ of all the 1-jets $j^1\psi$ of local diffeomorphisms of $\mathcal{B}$ induced from the elements of $\bar{\Omega}(\mathcal{B})$. It is not hard to prove that $\Omega(\mathcal{B})$ is a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$, provided that $\bar{\Omega}(\mathcal{B})$ be a Lie groupoid. We denote by α and β the source and target mappings which are in fact the restrictions of $\alpha, \beta : \Pi^1(\mathcal{B}, \mathcal{B}) \longrightarrow \mathcal{B}$. Moreover, the canonical projection

$$\lambda : \bar{\Omega}(\mathcal{B}) \longrightarrow \Omega(\mathcal{B}) \quad [j^1_{X,\psi(X)}\Psi] \mapsto j^1_{X,\psi(X)}\psi,$$

is a groupoid morphism.

We also consider the set $G(X_0)$ of the induced local isomorphisms from the elements of $\bar{G}(X_0)$; $G(X_0)$ is a group. Next, we put $\Omega_{X_0}(\mathcal{B}) = \beta^{-1}(X_0)$. Proceeding in a similar way than above, we can prove the following.

Proposition 10.7.
(i) *$G(X_0)$ is a Lie group.*
(ii) *$\Omega_{X_0}(\mathcal{B})$ is a principal bundle over $\mathcal{B}$ with structure group $G(X_0)$ and projection β.*

In fact, the sections $\tau_a : U_a \longrightarrow \Omega_{X_0}(\mathcal{B})$ defined by $\tau_a = \lambda \circ \sigma_a$ are adapted for $\Omega_{X_0}(\mathcal{B})$.

The following construction is also standard in the theory of G-structures and Lie groupoids [18,19,49].

Suppose that $\bar{Z}_0 = j^1_{e_1,\Phi(e_1)}\Phi \in \bar{F}^2\mathcal{B}$ is a non-holonomic frame of second order at X_0. (In particular, $\bar{Z}_0$ may be a holonomic frame.) Define a map $\bar{h} : \bar{G}(X_0) \longrightarrow \bar{G}^2(3)$, by $\bar{h}(\bar{Z}) = \bar{Z}_0^{-1} \cdot \bar{Z} \cdot \bar{Z}_0$. (To do the above jet composition we choose a representative of the equivalence class modulo $Gl(3, \mathbb{R})$, and the final result is independent of that choice.) Then $\bar{h}$ is differentiable and $\bar{G} = \bar{h}(\bar{G}(X_0))$ is a Lie subgroup of $\bar{G}^2(3)$. $\bar{G}$ is called the *isotropy group* of the Cosserat medium $\mathcal{B}$. It is uniquely defined up to conjugation (see Remark 10.13).

Next, let $\{U_a\}$ be the open covering obtained in the proof of Proposition 10.6. We can assume that $\sigma_{a_0}(X_0)$ is the identity of $\bar{G}(X_0)$ (if that is not the case, we define $\sigma'_a(X) = \sigma_a(X) \cdot \sigma_{a_0}(X_0)^{-1}$). For a point $X \in U_a \cap U_b$ we have $\sigma_b(X) = \sigma_a(X)\bar{g}_{ab}(X)$, where $\bar{g}_{ab}(X) \in \bar{G}(X_0)$. If we put $S_a(X) = \sigma_a(X) \cdot \bar{Z}_0$, we deduce that $S_a : U_a \longrightarrow \bar{F}^2\mathcal{B}$ and $S_b(X) = S_a(X) \cdot \bar{h}(\bar{g}_{ab}(X))$.

Therefore the family $\{U_a, S_a\}$ defines a second-order non-holonomic $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ on $\mathcal{B}$ with transition functions $\{\bar{h}(\bar{g}_{ab})\}$.

The principal bundles $\bar{\Omega}_{X_0}(\mathcal{B})$ and $\bar{\omega}_{\bar{G}}(\mathcal{B})$ are isomorphic.

Now, we put $\tilde{X}_0 = \pi^2_1(\bar{Z}_0)$. Hence $\tilde{X}_0$ is a linear frame at X_0. As above, we define a map $h : G(X_0) \longrightarrow Gl(3, \mathbb{R})$, by $h(Z) = Z_0^{-1} \cdot Z \cdot Z_0$. Then h is differentiable and $G = h(G(X_0))$ is a Lie subgroup of $Gl(3, \mathbb{R})$.

Also, let $\{\tau_a\}$ be the local sections obtained by projection from $\{\sigma_a\}$ (see Proposition 10.7). We have new local sections $\{T_a\}$ which define a G-structure $\omega_G(\mathcal{B})$ over $\mathcal{B}$ with transition functions $\{h(g_{ab})\}$. This G-structure is in fact the canonical projection of the second-order non-holonomic $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$.

A section S_a of $\bar{F}^2\mathcal{B}$ will be called a *local uniform reference*. If there exists a global section S of $\bar{F}^2\mathcal{B}$ it will be called a *uniform reference*. Notice that a (global) uniformity induces a global section S of $\bar{F}^2\mathcal{B}$.

If we suppose that the Cosserat continuum enjoys smooth global uniformity, then there exists a global uniformity $\sigma : \mathcal{B} \longrightarrow \bar{\Omega}_{X_0}(\mathcal{B})$ which induces a global section $S : \mathcal{B} \longrightarrow \bar{\omega}_{\bar{G}}(\mathcal{B})$. The second-order non-holonomic $\bar{G}$-structure is obtained by enlarging the global section S by means of $\bar{G}$. Of course, we have induced global sections $\tau : \mathcal{B} \longrightarrow \Omega_{X_0}(\mathcal{B})$ and $T : \mathcal{B} \longrightarrow \omega_G(\mathcal{B})$. Therefore, the projected G-structure is obtained by enlarging T by G.

Definition 10.8. A non-holonomic frame of second-order $\bar{Z}_0$ at X_0 will be called a *reference crystal.*

Summarizing the results we deduce that, associated with a uniform Cosserat continuum $\mathcal{B}$ there exist:
(i) a second-order non-holonomic $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ on $\mathcal{B}$;
(ii) a G-structure $\omega_G(\mathcal{B})$ on $\mathcal{B}$, obtained from $\bar{\omega}_{\bar{G}}(\mathcal{B})$ by projection, with structure group $G = pr_2(\bar{G})$.

Remark 10.9. Since the canonical projection $\bar{\pi}_1^2 : \bar{F}^2\mathcal{B} \longrightarrow F\mathcal{B}$ is a principal bundle homomorphism, then the $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ defines via the projection $\bar{\pi}_1^2$ a G'-structure $\omega'_{G'}(\mathcal{B})$ on $\mathcal{B}$, where $G' = pr_1(\bar{G})$. In fact, if we assume that $\mathcal{B}$ enjoys global smooth uniformity, then $\omega'_{G'}(\mathcal{B})$ is constructed by prolongating a global section $P : \mathcal{B} \longrightarrow F\mathcal{B}$ obtained by projecting the second-order non-holonomic parallelism S by means of the Lie group G'.

10.3. Homogeneous Cosserat media

As we have seen, a Cosserat continuum is uniform if the function W does not depend on the point X. In addition, a Cosserat continuum is said to be homogeneous if we can choose a global uniform reference which is constant on the body. In a more precise way, we introduce the following definition.

Definition 10.10. A Cosserat continuum $\mathcal{B}$ is said to be *homogeneous* with respect to a given reference crystal $\bar{Z}_0$ if it admits a global deformation $\tilde{\kappa}$, with an induced diffeomorphism κ on $\mathcal{B}$, such that $\bar{P} = \tilde{\kappa}^{-1}$ induces a uniform reference $\bar{P}$, i.e.,

$$\bar{P}(X) = j^1_{0,X}(\tilde{\kappa}^{-1} \circ F\tau_{\kappa(X)}), \quad \forall X \in \mathcal{B},$$

where $\tau_{\kappa(X)} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denotes the translation on $\mathbb{R}^3$ by the vector $\kappa(X)$ and $F\tau_{\kappa(X)}$ is the induced map. $\mathcal{B}$ is said to be *locally homogeneous* if every $X \in \mathcal{B}$ has a neighbourhood which is homogeneous. It is obvious that if $\mathcal{B}$ is homogeneous, then it is locally homogeneous.

We shall prove that this definition is independent on the choice of reference configuration. We also study what happen if we change the reference crystal.

Theorem 10.11. *If $\mathcal{B}$ is homogeneous then $\bar{\omega}_{\bar{G}}(F\mathcal{B})$ is an integrable prolongation. Hence $\omega_G(\mathcal{B})$ is also integrable. Conversely, if $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is an integrable prolongation then $\mathcal{B}$ is locally homogeneous.*

Proof. Assume that $\mathcal{B}$ is homogeneous. Hence, there exists a global deformation $\tilde{\kappa}$ which may be used in order to define a global uniform reference S. If we take local coordinates (x^i) on $\mathcal{B}$ given by the induced diffeomorphism κ, we deduce that S is locally expressed by

$$S(x^i) = \left(x^i, P^i_j(x), \delta^i_j, \frac{\partial P^i_j}{\partial x^k}\right),$$

where $P(x^i) = (x^i, P^i_j(x))$ is the local expression of the linear parallelism P. Therefore $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is an integrable prolongation.

Conversely, if $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is an integrable prolongation, then there exists a local adapted section S around each point of $\mathcal{B}$ which is an integrable prolongation. Thus, we can choose local coordinates (x^i) such that $S(x^i) = (x^i, P^i_j(x), \delta^i_j, \partial P^i_j/\partial x^k)$. Hence, we can take a local deformation $\tilde{\kappa}$ defined by $\tilde{\kappa}(x^i, x^i_j) = (x^i, P^i_k x^k_j)$, which implies the local homogeneity of $\mathcal{B}$. □

Remark 10.12. Notice that if $\mathcal{B}$ is homogeneous, then the macromedium is also homogeneous. Obviously, the converse is not true. In fact, the integrability of the G-structure $\omega_G(\mathcal{B})$ does not imply the prolongability of $\bar{\omega}_{\bar{G}}(\mathcal{B})$. We also notice that the homogeneity of a Cosserat medium $\mathcal{B}$ does not imply the integrability of the G'-structure $\omega'_{G'}(\mathcal{B})$.

Remark 10.13. (1) If we change the point X_0 to another point X'_0, then we obtain an isomorphic $\bar{G}$-structure. In fact, we take a local uniformity $\mathcal{S}$ joinning X_0 and X'_0 and, next, a crystal reference obtained by composing $\bar{Z}_0$ with $\mathcal{S}$.

(2) We have fixed a reference configuration Φ_0. Suppose that Φ_1 is another reference configuration such that the change of configuration is given by $\Psi = \Phi_1^{-1} \circ \Phi_0$. Therefore, by using Φ_1, the change of reference configuration Ψ yields an isomorphism between the respective $\bar{G}$-structures, provided that the reference crystal $\bar{Z}_0$ at X_0 is transported via Ψ to a reference crystal $j^1_{0,\psi(\upsilon(0))}(\Psi \circ \Upsilon)$, where $\bar{Z}_0 = j^1_{0,\upsilon(0)}\Upsilon$. Hence, the homogeneity is indifferent to a change of reference configuration. By the way, observe that the isotropy group $\bar{G}$ remains the same.

(3) Finally, suppose that we change the reference crystal $\bar{Z}_0$ to another reference crystal $\bar{Z}'_0$. In other words, we choose another non-holonomic second-order frame $\bar{Z}'_0$ at X_0. Hence we get $S'_a(X) = \sigma_a \cdot \bar{Z}'_0 = \sigma_a(X) \cdot \bar{Z}_0 \cdot (A, B, C)$, since $\bar{Z}'_0 = \bar{Z}_0 \cdot (A, B, C)$, $(A, B, C) \in \bar{G}^2(3)$. We deduce that the new $\bar{G}'$-structure is conjugate to the original $\bar{G}$ structure, and the isotropy groups $\bar{G}'$ and $\bar{G}$ are conjugate, namely

$$\bar{G}' = (A, B, C)\bar{G}(A, B, C)^{-1}, \qquad \bar{\omega}_{\bar{G}'}(\mathcal{B}) = \bar{\omega}_{\bar{G}}(\mathcal{B})(A, B, C).$$

As we know, if one first-order G-structure is integrable, the same holds for all conjugate G-structures. However, if a $\bar{G}$-structure is integrable (or an integrable prolongation), a conjugate $\bar{G}'$-structure may fail to be also integrable (or an integrable prolongation). We can easily check this fact by considering, for instance, an integrable non-holonomic parallelism. Our present definition of homogeneity is given with respect to a fixed reference crystal. Indeed, if we change from a reference crystal $\bar{Z}_0$ to another $\bar{Z}'_0$ then the homogeneity does not hold, in general.

11. Cosserat media with global uniformity

Along this section we shall suppose that $\mathcal{B}$ enjoys smooth global uniformity. This means that there exists a global uniformity σ which induces a global uniform reference S, i.e., a second-order non-holonomic parallelism on $\mathcal{B}$.

Then we have the following parallelisms:

(i) a second-order non-holonomic parallelism $S : \mathcal{B} \longrightarrow \bar{F}^2\mathcal{B}$ on $\mathcal{B}$;

(ii) a linear parallelism $P : \mathcal{B} \longrightarrow F\mathcal{B}$ on $\mathcal{B}$ defined by the projection of S, namely $P = \bar{\pi}_1^2 \circ S$;

(iii) a linear parallelism $Q : \mathcal{B} \longrightarrow F\mathcal{B}$ on $\mathcal{B}$ defined from the "underlying uniformity", namely $Q = \tilde{\pi}_1^2 \circ S$.

Of course, S is semi-holonomic if and only if $Q = P$.

Notice that the $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$, the G'-structure $\omega'_{G'}(\mathcal{B})$ and the G-structure $\omega_G(\mathcal{B})$ are obtained by enlarging the corresponding global sections S, P and Q, by the Lie groups $\bar{G}$, G' and G, respectively.

For a point $X \in \mathcal{B}$, $P(X)$ and $Q(X)$ are linear frames at X on $\mathcal{B}$ and $S(X)$ is a non-holonomic frame of second order at X. In local coordinates we have

$$P(x^i) = (x^i, P_a^i(x^r)), \qquad Q(x^i) = (x^i, Q_a^i(x^r)),$$
$$S(x^i) = (x^i, S_j^i(x^r), S^i_{,j}(x^r), S^i_{j,k}(x^r)),$$

where $S_j^i = P_j^i$ and $S^i_{,j} = Q_j^i$.

From now on, we shall adopt the following notation:

$$S(x^i) = (x^i, P_j^i, Q_j^i, R_{jk}^i),$$

where $R_{jk}^i = S^i_{j,k}$.

The parallelism P determines three linearly independent vector fields $\{P_1, P_2, P_3\}$ on $\mathcal{B}$ which can be locally expressed as

$$P_a = P_a^i \frac{\partial}{\partial x^i},$$

P defines a linear connection Γ_1 whose Christoffel components in a coordinate system (x^i) on $\mathcal{B}$ are:

$$(\Gamma_1)_{jk}^i = -(P^{-1})_k^l \frac{\partial P_l^i}{\partial x^j}.$$

The linear connection Γ_1 has torsion T_1 but no curvature. We notice that the connection Γ_1 is an adapted connection to the parallelism defined by P, and, hence, it is adapted to the G'-structure $\omega'_{G'}(\mathcal{B})$. Furthermore, we know that P is integrable if and only if Γ_1 is locally flat.

In a similar way, the parallelism Q determines three linearly independent vector fields $\{Q_1, Q_2, Q_3\}$ on $\mathcal{B}$ which can be locally expressed as

$$Q_a = Q_a^i \frac{\partial}{\partial x^i},$$

Q defines a linear connection Γ_2 whose Christoffel components are:

$$(\Gamma_2)^i_{jk} = -(Q^{-1})^l_k \frac{\partial Q^i_l}{\partial x^j}.$$

As above, Γ_2 is an adapted flat connection to the parallelism defined by Q and, it is also adapted to the G-structure $\omega_G(\mathcal{B})$. As we know, Q is integrable if and only if Γ_2 is locally flat. The torsion tensor of Γ_2 will be denoted by T_2.

According to Section 7.1, S induces a global invariant section $\tilde{S}: F\mathcal{B} \longrightarrow \bar{F}^2\mathcal{B}$ and, hence, a third linear connection Γ_3 whose Christoffel components are:

$$(\Gamma_3)^i_{jk} = -R^i_{rs}(P^{-1})^r_k(Q^{-1})^s_j.$$

Consider the difference tensor D of the two connections Γ_1 and Γ_3, i.e., $D = \nabla_1 - \nabla_3$, where ∇_1 and ∇_3 are the covariant derivatives of Γ_1 and Γ_3, respectively. T_2 and D will be called the *inhomogeneity tensors*.

The geometric characterization of the local homogeneity is as follows.

First, we consider the case of Cosserat media without symmetries, i.e., the Lie group $\bar{G}$ is trivial, $\bar{G} = \{(1, 1, 0)\}$. In that case, the $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is a second-order non-holonomic parallelism S on $\mathcal{B}$. As a consequence, the G-structure $\omega_G(\mathcal{B})$ and the G'-structure $\omega'_{G'}(\mathcal{B})$ are ordinary parallelisms on $\mathcal{B}$, which will be denoted by P and Q, as above. From Theorem 10.11 and Corollary 7.8 we obtain the following result:

Theorem 11.1. *$\mathcal{B}$ is locally homogeneous if and only if the inhomogeneity tensors identically vanish, i.e., $T_2 = 0$ and $D = 0$.*

Remark 11.2. Notice that a section S of $\bar{\pi}^2: \bar{F}^2\mathcal{B} \longrightarrow \mathcal{B}$ may be valued into $\hat{F}^2\mathcal{B}$ or $F^2\mathcal{B}$. But if the symmetry group $\bar{G}$ is not semi-holonomic neither holonomic, then the $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is a genuine non-holonomic structure.

If the isotropy group $\bar{G}$ is not trivial, we deduce from Theorem 10.11 and Corollary 7.8 the following result:

Theorem 11.3. *$\mathcal{B}$ is locally homogeneous if and only if there exists an adapted local section on which T_2 and D are identically zero.*

12. A classification of Cosserat media

In this section we shall consider two particular cases of Cosserat media.

First of all, we shall give an alternative description of the constitutive Eq. (8).

Notice that a 1-jet $j^1_{X,\kappa(X)}\tilde{\kappa}$ may be represented as a triple (p, q, r), where

$$p^i_j = \kappa^i_j(x^a), \quad q^i_j = \frac{\partial \kappa^i}{\partial r^j}(x^a), \quad r^i_{jk} = \frac{\partial \kappa^i_j}{\partial r^k}(x^a),$$

with $\tilde{\kappa}(x^i, x^i_j) \equiv (\kappa^a(x^i), \kappa^a_b(x^i))$, $1 \le i, j, k, a, b \le 3$.

Therefore, we can write the constitutive equation as follows:

$$W = W(p, q, r; X), \tag{11}$$

i.e., $W = W(p(X), q(X), r(X))$, where $p(X) = \varphi(X)$, $q(X) = (\nabla\kappa)(X)$, and $r(X) = (\nabla p)(X)$, $\varphi = (\kappa^i_j)$. We are using here a slight different notation in order to connect with the usual notations in Continuun Mechanics (see [69,80,81], for instance). There the dependence on the point in $\mathcal{B}$ is explicitly indicated, but this dependence automatically appears if we use a jet formulation.

We can distinguish three different kinds of Cosserat media:

(i) *Holonomic Cosserat media*. They are defined by the condition

$$\kappa^i_j = \frac{\partial\kappa^i}{\partial x^j},$$

at every point $X \in \mathcal{B}$. We then have

$$\frac{\partial\kappa^i_j}{\partial x^k} = \frac{\partial^2\kappa^i}{\partial x^j\partial x^k},$$

and the constitutive equation becomes

$$W = W\left(\frac{\partial\kappa^i}{\partial x^j}, \frac{\partial^2\kappa^i}{\partial x^j\partial x^k}\right),$$

i.e., we are in presence of a material of second grade:

$$W = W(p, \nabla p; X). \tag{12}$$

For the sake of consistence of the constitutive equations, the admissible uniformities must be of the same kind, and, therefore the material symmetry group is actually a Lie subgroup of the second-order holonomic group $G^2(3)$.

(ii) *Semi-holonomic Cosserat media*. They are defined by the condition

$$\kappa^i_j = \frac{\partial\kappa^i}{\partial x^j},$$

only at the point X. Hence,

$$\frac{\partial\kappa^i_j}{\partial x^k}(Y) \neq \frac{\partial^2\kappa^i}{\partial x^j\partial x^k}(Y)$$

for all points $Y \neq X$, and the constituive equation becomes

$$W = W\left(\kappa^i_j, \frac{\partial\kappa^i_j}{\partial x}\right),$$

i.e.,

$$W = W(p, \nabla p; X). \tag{13}$$

Eqs. (12) and (13) are apparently the same. In spite of that, note that the meanings of p and ∇p in both equations are completely different. In fact, Eq. (13) means that W *does not depend on the macromedium.*

As above, for the sake of consistency, the admissible uniformities must be of the same nature, and the material symmetry group is actually a Lie subgroup of the second-order semi-holonomic group $\hat{G}^2(3)$.

(iii) *Strictly non-holonomic Cosserat media.* They are defined without conditions.

12.1. Homogeneity of semi-holonomic Cosserat media

In this case, the second-order non-holonomic $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is in fact a reduction of the second-order semi-holonomic frame bundle $\hat{F}^2(\mathcal{B})$, i.e., $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is a second-order semi-holonomic structure, provided that we have chosen a semi-holonomic reference crystal. Thus, $\bar{\omega}_{\bar{G}}(\mathcal{B}) \subset \hat{F}^2(\mathcal{B})$ and $\bar{G} \subset \hat{G}^2(3)$. In such a case we shall use the notation $\hat{\omega}_{\hat{G}}(\mathcal{B})$ for the reduced bundle and $\hat{G}$ for the structure group.

If we suppose that $\mathcal{B}$ enjoys smooth global uniformity, we deduce that there exists a second-order semi-holonomic parallelism $S : \mathcal{B} \longrightarrow \hat{F}^2\mathcal{B}$. The induced global parallelisms $P : \mathcal{B} \longrightarrow F\mathcal{B}$ and $Q : \mathcal{B} \longrightarrow F\mathcal{B}$ coincide.

We can write in local coordinates $S(x^i) = (x^i, P^i_j(x^r), R^i_{jk}(x^r))$ and $P(x^i) = (x^i, P^i_j(x^r))$.

We deduce that the two linear connections Γ_1 and Γ_2 are the same, namely $\Gamma = \Gamma_1 = \Gamma_2$.

Consider again the diference tensor D of the two connections Γ and Γ_3, i.e., $D = \nabla - \nabla_3$, where ∇ and ∇_3 are the covariant derivatives of Γ and Γ_3, respectively. Remember that Γ is the linear connection defined from the projected parallelism P. Its torsion tensor will be denoted as above by T, and, the two tensors T and D will be called the *inhomogeneity tensors.*

Now, Theorems 11.1 and 11.3 have the same form:

Theorem 12.1.
(1) *If the isotropy group is trivial, then $\mathcal{B}$ is locally homogeneous if and only if the inhomogeneity tensors T and D simultaneously vanish, i.e., $T = 0$ and $D = 0$.*
(2) *In the general case, $\mathcal{B}$ is locally homogeneous if and only if there exists an adapted local section on which the inhomogeneity tensors simultaneously vanish.*

12.2. Homogeneity of bodies of second grade

In this case, all the configurations $\tilde{\kappa}$ and the local isomorphisms given by the uniformity property of the Cosserat medium $\mathcal{B}$ are natural prolongations to the frame bundle of the induced diffeomorphisms on the basis, i.e., $\tilde{\kappa} = F\kappa$ and $\Psi = F\psi$, and the response functional W may be written as follows:

$$W = W(j^2_{X,\kappa(X)}\kappa),$$

since $\tilde{\kappa} = F(\kappa)$, or, equivalently,

$$W = W(F, \nabla F; X),$$

where $F = \nabla\kappa$. Therefore, we are in presence of a material body of second grade (see [17–19]).

Furthermore, if we choose a second-order frame Z_0 at a point X_0 as above, we obtain a second-order $\check{G}$-structure $\check{\omega}_{\check{G}}(\mathcal{B})$ on $\mathcal{B}$. If we suppose that $\mathcal{B}$ enjoys smooth global uniformity, we deduce that there exists a second-order parallelism $S : \mathcal{B} \longrightarrow \check{F}^2\mathcal{B}$. We can write in local coordinates $S(x^i) = (x^i, P^i_j(x^r), R^i_{jk}(x^r))$, where $R^i_{jk} = R^i_{kj}$.

The induced global parallelism $P : \mathcal{B} \longrightarrow F\mathcal{B}$ is given by $P(x^i) = (x^i, P^i_j(x^r))$, and, as in the previous case, we have $P = Q$.

Now, the connection Λ induced by S is symmetric and $T = 0$. We can consider the difference tensor D of the two connections Γ and Γ_3, i.e., $D = \nabla - \nabla_3$, where ∇ and ∇_3 are the covariant derivatives of Γ and Γ_3, respectively. Remenber that Γ is the linear connection defined from the projected parallelism P. Since Γ is symmetric we deduce that

$$T = 0, \;\; D = 0 \Longleftrightarrow D = 0.$$

We call D the *inhomogeneity* tensor. Now, Theorems 11.1 and 11.3 read as follows:

Theorem 12.2.
(1) *If the isotropy group is trivial, then $\mathcal{B}$ is locally homogeneous if and only if the inhomogeneity tensor D vanishes.*
(2) *In the general case, $\mathcal{B}$ is locally homogeneous if and only if there exists an adapted local section on which the inhomogeneity tensor vanishes.*

12.3. More about homogeneous Cosserat media

In Definition 10.11 we have introduced a notion of homogeneity with respect to a given reference crystal. We now give a general notion of homogeneity.

Definition 12.3. A Cosserat medium $\mathcal{B}$ is said to be *(locally) homogeneous* if it is (locally) homogeneous with respect to some reference crystal.

Consider now a change of reference crystal. This means that we choose another non-holonomic frame of second-order $\bar{Z}'_0$ at the point X_0. Hence, we have $\bar{Z}'_0 = \bar{Z}_0(A, B, C)$, where $(A, B, C) \in \bar{G}^2(3)$. Therefore, the new second-order non-holonomic parallelism S' is given by $S' = S(A, B, C)$, where S is the second-order non-holonomic parallelism obtained from $\bar{Z}_0$. We obtain

$$S'(x^i) = (x^i, P^i_a A^a_j, Q^i_a B^a_j, P^i_a C^a_{jk} + R^i_{ab} A^a_j B^b_k).$$

A direct computation shows that the Christoffel components of the new three linear connections Γ'_1, Γ'_2 and Γ'_3, are:

$$(\Gamma_1')^i_{jk} = (\Gamma_1)^i_{jk}, \quad (\Gamma_2')^i_{jk} = (\Gamma_2)^i_{jk},$$
$$(\Gamma_3')^i_{jk} = (\Gamma_3)^i_{jk} - P^i_a C^a_{rs} (A^{-1})^r_t (P^{-1})^t_k (B^{-1})^s_c (Q^{-1})^c_j,$$

where $(\Gamma_1)^i_{jk}$, $(\Gamma_2)^i_{jk}$, and $(\Gamma_3)^i_{jk}$ are the Christoffel components of the three linear connections induced from S.

From these expressions we obtain

$$T_2' = T_2, \tag{14}$$
$$(D')^i_{jk} = D^i_{jk} + P^i_a C^a_{rs} (A^{-1})^r_t (P^{-1})^t_k (B^{-1})^s_c (Q^{-1})^c_j. \tag{15}$$

If $T_2 = 0$ and $D = 0$, we know that $\mathcal{B}$ is locally homogeneous. From (14) we deduce that $T_2 = 0$ if and only if $T_2' = 0$. If $D \neq 0$, then the Cosserat medium $\mathcal{B}$ is not locally homogeneous with respect to $\bar{Z}_0$, but we can search for a change of reference crystal on which $D' = 0$, and, hence, $\mathcal{B}$ would be locally homogeneous with respect to that new reference crystal.

We have

$$\begin{aligned}
D' = 0 &\Longleftrightarrow D^i_{jk} = -P^i_a C^a_{rs} (A^{-1})^r_t (P^{-1})^t_k (B^{-1})^s_c (Q^{-1})^c_j \\
&\Longleftrightarrow D^u_{vw} (P^{-1})^i_u P^w_j Q^r_k = \sigma^i_{jk}, \\
&\qquad \text{(where } \sigma^i_{jk} = -C^i_{vw} (A^{-1})^v_j (B^{-1})^w_k = \text{constant)} \\
&\Longleftrightarrow D(Q_j, P_k) = \sigma^u_{kj} P_u, \quad \text{with the } \sigma\text{'s constant} \\
&\Longleftrightarrow \nabla_1 D_{jk} = 0, \quad \text{where } D_{jk} = D(Q_j, P_k),
\end{aligned}$$

where ∇_1 denotes the covariant derivative defined by Γ_1. Here $D_{jk} = D(Q_j, P_k)$ are not the components of any tensor. In fact, D is a tensor field of type $(1, 2)$ and D_{jk} is the vector field obtained by applying D to the two vector fields Q_j and P_k. Thus, $\nabla_1 D_{jk}$ are 1-forms.

Thus, in order to obtain a new reference crystal with respect to which $\mathcal{B}$ would be locally homogeneous, we can proceed as follows. First, we compute the nine covariant derivatives $\nabla_1 D_{jk}$. If they simultaneously vanish, we take the reference crystal $\bar{Z}_0 = \bar{Z}_0(A, B, C)$, where

$$\sigma^i_{jk} = -C^i_{vw} (A^{-1})^v_j (B^{-1})^w_k,$$

being

$$D(Q_j, P_k) = \sigma^u_{kj} P_u.$$

There exist, of course, many possible choices. From the above discussion, we deduce that $D' = 0$ and we conclude that $\mathcal{B}$ is locally homogeneous with respect to $\bar{Z}_0'$.

Consider a change of configuration $\tilde{\kappa}(x^i, x^i_j) = (\kappa^i(x^a), \kappa^i_k(x^a) x^k_j)$. The second-order non-holonomic parallelism S' defined by using that new configuration is given by

$$S'(x^i) = \left(x^i, \kappa^i_a P^a_j, \frac{\partial \kappa^i}{\partial x^a} Q^a_j, \frac{\partial \kappa^i_a}{\partial x^b} Q^b_k P^a_j + \kappa^i_a R^a_{jk} \right). \tag{16}$$

Now, suppose that $\mathcal{B}$ is locally homogeneous, or, equivalently, $T_2 = 0$ and $D = 0$ in the first configuration. Hence, we can choose local coordinates around each point in $\mathcal{B}$ such that

$$Q^i_j = \delta^i_j, \qquad R^i_{jk} = \frac{\partial P^i_j}{\partial x^k}$$

(see Section 7.4). Thus, we have

$$S(x^i) = \left(P^i_j, \delta^i_j, \frac{\partial P^i_j}{\partial x^k}\right),$$

i.e., S is an integrable prolongation of P. Next, we perform the change of configuration $(x^i, x^i_j) \rightsquigarrow (x^i, (P^{-1})^i_k x^k_j)$. From (16) we obtain

$$S'(X) = (X, 1, 1, 0), \tag{17}$$

or, in other words, we have found a configuration on which S have constant components. Observe that Eq. (17) means that the first and third matrices in S' with respect to the basis $\{P_1, P_2, P_3\}$ are (1) and (0) and, the the second matrix in S' is (1) with respect to the local coordinates (x^i).

Conversely, let us suppose that there exists a configuration on which S has constant components, namely $S(X) = (P(X), Q(X), R(X))$, where $P(X)$, $Q(X)$ and $R(X)$ are constant. If $P(X) = A$, $Q(X) = B$ and $R(X) = C$, where $(A, B, C) \in \bar{G}^2(3)$, then we can perform a change of reference crystal by means of $(A, B, C)^{-1}$ such that the new non-holonomic second-order parallelism is $S'(X) = (1, 1, 0)$. Consider an arbitrary change of reference configuration $\tilde{\kappa}(x^i, x^i_j) = (\kappa(x^i), \kappa^i_k(x^u)x^k_j)$. With respect to the new reference configuration we have

$$S''(x^i) = \left(x^i, \kappa^i_j, \frac{\partial \kappa^i}{\partial x^j}, \frac{\partial \kappa^i_j}{\partial x^k}\right),$$

which shows that $\mathcal{B}$ is, in fact, locally homogeneous.

Summarizing the above discussion, in order to check the local homogeneity of a Cosserat medium, we have to pick an arbitrary adapted section and compute the two tensors D and T_2. If $T_2 \neq 0$, the material is not homogeneous. If $T_2 = 0$, but $D \neq 0$ we have a chance. In fact, we must compute the nine covariant derivatives $\nabla_1 D_{jk}$. If all them vanish, we can perform a change of reference crystal in order to obtain an homogeneous configuration. Of course, this discussion holds when the isotropy group is trivial. If the isotropy group is continuous (even not trivial) we have an additional degree of freedom. Thus, in order to decide about the local homogeneity, we must consider the existence of alternative adapted sections on which the inhomogeneity tensors would vanish. As in the case of simple materials, we can obtain in some cases a complete answer by using geometrical results on the prolongability of second-order non-holonomic $\bar{G}$-structures (see [37] and Section 13).

Remark 12.4. It is important to distinguish between changes of coordinates and changes of configurations. For simple media, there are no mathematical differences, since a change of coordinates is a local diffeomorphism which can be interpreted as a local change of configuration, and conversely. However, for Cosserat media, there is a subtle difference. In fact, a deformation is a morphism of principal bundles, but not every morphism of frame bundles is of the form $\tilde{\kappa} = F\kappa$. This situation occurs only in the case of holonomic Cosserat media. Thus, the existence of a constant uniform reference S does not imply that S is integrable, it only implies that S is an integrable prolongation.

13. Homogeneity of particular Cosserat media

Throughout this section we shall consider local homogeneity with respect to a fixed reference crystal (see Section 12.3).

13.1. Cosserat–Toupin media

We call a second-order non-holonomic $\bar{G}$-structure on $\mathcal{B}$ a *Cosserat–Toupin structure* when the structure group of the Cosserat medium is a Toupin subgroup.

We put $\bar{G} = (G_1, G_2, \alpha(G_1, G_2) - G_1\alpha)$, where G_1 and G_2 are Lie subgroups of $Gl(3, \mathbb{R})$ and α is a given element of $\mathcal{B}^2(3)$. Since $\bar{G}$ is the conjugate subgroup of the subgroup $(G_1, G_2, 0)$, then the $\bar{G}$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ is conjugate to the $(G_1, G_2, 0)$-structure $\bar{\omega}$. We then only consider the case $(G_1, G_2, 0)$.

Notice that there exist two projected G-structures, namely, a G_1-structure ω_1 obtained by enlarging P by means of G_1 and, a G_2-structure ω_2 obtained by enlarging Q by means of G_2.

Recall that $\bar{\omega}$ is defined by enlarging the global section $S : \mathcal{B} \longrightarrow \bar{F}^2\mathcal{B}$ to the whole group $(G_1, G_2, 0)$. Denote by Γ_1, Γ_2 and Γ_3 the three linear connections introduced in Section 13.

Suppose that $\mathcal{B}$ is locally homogeneous. Hence, $\bar{\omega}$ is an integrable prolongation and, then there exist local coordinates (x^i) and a local section s locally expressed by

$$s(x^i) = \left(x^i, p^i_j, \delta^i_j, \frac{\partial p^i_j}{\partial x^k}\right).$$

Therefore, we obtain $S(x^i) = (x^i, P^i_j, Q^i_j, R^i_{jk}) = (x^i, p^i_j, \delta^i_j, \partial p^i_j/\partial x^k)(A, B, 0)$, where $A \in G_1$, $B \in G_2$, which implies

$$P^i_j = p^i_a A^a_j, \qquad Q^i_j = B^i_j, \qquad R^i_{jk} = \frac{\partial p^i_a}{\partial x^b} A^a_j B^b_k.$$

A direct computation yields:

$$(\Gamma_1)^i_{jk} = -(p^{-1})^a_k \frac{\partial p^i_a}{\partial x^j} - (A^{-1})^u_r \frac{\partial A^t_u}{\partial x^j} (p^{-1})^r_k p^i_t, \tag{18}$$

$$(\Gamma_2)^i_{jk} = -(B^{-1})^a_k \frac{\partial B^i_a}{\partial x^j}, \tag{19}$$

$$(\Gamma_3)^i_{jk} = (p^{-1})^u_k \frac{\partial p^i_u}{\partial x^j}. \tag{20}$$

From (20) we deduce that Γ_3 coincides in the coordinate neighbourhood with the flat connection defined by the local parallelism $p(x^i) = (x^i, p^i_j(x))$. Therefore, Γ_3 is a flat G_1-connection.

Moreover, ω_2 is integrable, i.e., the macromedium is locally homogeneous.

Conversely, suppose that the macromedium is locally homogeneous, i.e., ω_2 is integrable and Γ_3 is a G_1-connection.

Since ω_2 is integrable, then there exist local coordinates (x^i) on $\mathcal{B}$ such that $\alpha(x^i) = (x^i, 1) \in \omega_2$. Hence, we have $S(x^i) = (x^i, P^i_j, Q^i_j, R^i_{jk})$, where $(x^i, Q^i_j) = (x^i, 1)B = (x^i, B^i_j)$ for some $B \in G_2$. Thus, we have that $(Q^i_j = B^i_j) \in G_2$. We construct a local section

$$s(x^i) = S(x^i)(1, B^{-1}, 0) = (x^i, P^i_j, 1, R^i_{js}(Q^{-1})^s_k),$$

which is also adapted to $\bar{\omega}$ since $(1, B^{-1}, 0) \in (G_1, G_2, 0)$. The point now is to find a local section $\sigma(x^i) = (x^i, p^i_j, 1, r^i_{jk})$ such that $s(x^i) = \sigma(x^i)(A, 1, 0)$, where $A(x^i) \in G_1$, $r^i_{jk} = (\partial p^i_j)/(\partial x^k)$.

If such a section exists, then we have

$$P^i_j = p^i_a A^a_j, \qquad R^i_{js}(Q^{-1})^s_k = r^i_{ak} A^a_j. \tag{21}$$

From (21) and by a direct computation, we deduce that

$$(A^{-1})^i_a \frac{\partial A^a_j}{\partial x^k} = (P^{-1})^i_a \frac{\partial P^a_j}{\partial x^k} = R^a_{js}(P^{-1})^i_a (Q^{-1})^s_k. \tag{22}$$

Since Γ_1 and Γ_3 are G_1-connections we deduce that the right-hand side of Eq. (22) belongs to the Lie algebra $\mathfrak{g}_1$ of G_1. Therefore, Eq. (22) has a solution A in G_1 and we are able to construct the required section σ.

Thus, we have proved the following:

Theorem 13.1. *If $\mathcal{B}$ is locally homogeneous, then the macromedium is also locally homegeneous and Γ_3 is a flat G_1-connection. Conversely, if the macromedium is locally homogeneous and Γ_3 is a G_1-connection, then $\mathcal{B}$ is locally homogeneous.*

Assume that $\mathcal{B}$ is a holonomic or semi-holonomic Cosserat medium. This means that $G_1 = G_2 = G$, $P = Q$ and $\Gamma_1 = \Gamma_2$. In that case, Theorem 13.1 reads as follows:

Corollary 13.2. *Let $\mathcal{B}$ be a holonomic or semi-holonomic Cosserat medium. Then, if $\mathcal{B}$ is locally homogeneous, the macromedium is also locally homegeneous and Γ_3 is a locally flat G-connection. Conversely, if the macromedium is locally homogeneous and Γ_3 is a G-connection, then $\mathcal{B}$ is locally homogeneous.*

13.2. $(1, 1, \Sigma_{(1,1)})$-structures

Suppose that the isotropy group is $\bar{G} = (1, 1, \Sigma_{(1,1)})$.

Consider a $(1, 1, \Sigma_{(1,1)})$-structure $\bar{\omega}_{\bar{G}}(\mathcal{B})$ on $\mathcal{B}$. In that case, $G = G' = \{1\}$, or, in other words, the induced G-structure $\omega_G(\mathcal{B})$ is the linear parallelism Q and the induced G'-structure $\omega'_{G'}(\mathcal{B})$ is the linear parallelism P.

Denote by Γ_1, Γ_2 and Γ_3 the three linear connections defined from the second-order non-holonomic parallelism S. We can define for each $\alpha \in \Sigma_{(1,1)}$ a linear connection on $\mathcal{B}$ as follows. Consider the global section $S_\alpha = S(1, 1, \alpha)$, i.e.,

$$\begin{aligned} S_\alpha(x) &= (x, P^i_j(x), Q^i_j(x), R^i_{jk}(x))(1, 1, \alpha) \\ &= (x, P^i_j(x), Q^i_j(x), R^i_{jk}(x) + P^i_r(x)\alpha^r_{jk}). \end{aligned}$$

Of course, if we consider a function $\alpha : \mathcal{B} \longrightarrow \Sigma_{(1,1)}$ we can also define a section S_α as above.

The global section S_α determines a new linear connection $\Gamma_{3,\alpha}$ which in an arbitrary system of coordinates has Christoffel components

$$(\Gamma_{3,\alpha})^i_{jk} = -(R^i_{rs} + P^i_a\alpha^a_{rs})(P^{-1})^r_k(Q^{-1})^s_j = (\Gamma_3)^i_{jk} - P^i_a\alpha^a_{rs}(P^{-1})^r_k(Q^{-1})^s_j.$$

Notice that $\Gamma_{3,0} = \Gamma_3$.

Now, suppose that $\mathcal{B}$ is locally homogeneous. Hence, there exists an adapted section s which is an integrable prolongation. This means that there exist local coordinates (x^i) around each point of $\mathcal{B}$ such that

$$s(x^i) = \left(x^i, p^i_j, \delta^i_j, \frac{\partial p^i_j}{\partial x^k}\right)$$

is an adapted section. Hence, we have

$$\begin{aligned} S(x^i) = (x^i, P^i_j, Q^i_j, R^i_{jk}) &= \left(x^i, p, 1, \frac{\partial p^i_j}{\partial x^k}\right)(1, 1, \alpha(x^a)) \\ &= \left(x^i, p^i_j, 1, p^i_a\alpha^a_{jk} + \frac{\partial p^i_j}{\partial x^k}\right), \end{aligned}$$

for some element $\alpha(x^a) \in \Sigma_{(1,1)}$.

Consequently, we obtain

$$P^i_j = p^i_j, \qquad Q^i_j = \delta^i_j, \qquad R^i_{jk} = p^i_a\alpha^a_{jk} + \frac{\partial p^i_j}{\partial x^k}.$$

Therefore, the Christoffel components of the three linear connections Γ_1, Γ_2 and Γ_3 in these coordinates are the following:

$$(\Gamma_1)^i_{jk} = -(p^{-1})^a_k \frac{\partial p^i_a}{\partial x^j}, \tag{23}$$

$$(\Gamma_2)^i_{jk} = 0, \tag{24}$$

$$(\Gamma_3)^i_{jk} = -(p^{-1})^a_k \frac{\partial p^i_a}{\partial x^j} - p^i_a\alpha^a_{rj}(p^{-1})^r_k. \tag{25}$$

From (25) we deduce that

$$(\Gamma_3)^i_{jk} = (\Gamma_1)^i_{jk} - p^i_a \alpha^a_{rj} (p^{-1})^r_k, \tag{26}$$

from which we have that $\Gamma_{3,-\alpha} = \Gamma_1$ on the domain of the local coordinates (x^i).

By the way, observe that the macromedium is locally homogeneous since the torsion tensor T_2 of Γ_2 vanishes, which is equivalent to the integrability of the linear parallelism Q.

Conversely, suppose that the macromedium is locally homogeneous and that there exists a function $\alpha : \mathcal{B} \longrightarrow \Sigma_{(1,1)}$ such that $\Gamma_{3,\alpha} = \Gamma_1$.

Since $\omega_G(\mathcal{B})$ is integrable, i.e., Q is integrable, then there exist local coordinates (x^i) around each point of $\mathcal{B}$ such that the local section $s(x^i) = (x^i, 1)$ is adapted to $\omega_G(\mathcal{B})$, or, equivalently, $Q(x^i) = 1$. Thus, we obtain $S(x^i) = (x^i, P^i_j, 1, R^i_{jk})$.

Next, since $\Gamma_{3,\alpha} = \Gamma_1$ we obtain

$$R^i_{jr} + P^i_a \alpha^a_{rj} = \frac{\partial P^i_r}{\partial x^j}.$$

Therefore, the section S_α is an integrable prolongation, since

$$S_\alpha(x^i) = (x^i, P^i_j, 1, R^i_{jk} + P^i_a \alpha^a_{jk}).$$

We can summarize these results in the following theorem.

Theorem 13.3. *Suppose that $\mathcal{B}$ is locally homogeneous. Then the macromedium is locally homogeneous. Moreover, there exists a local coordinate system (x^i) around each point of $\mathcal{B}$ and a local function $\alpha(x^i)$ taking values into $\Sigma_{(1,1)}$ such that $(\Gamma_{3,-\alpha})^i_{jk} = (\Gamma_1)^i_{jk}$, or, in other words, the connections $\Gamma_{3,-\alpha}$ and Γ_1 coincide on the domain of the coordinates (x^i). Conversely, if the macromedium is locally homogeneous and there exists a function $\alpha : \mathcal{B} \longrightarrow \Sigma_{(1,1)}$ such that $\Gamma_{3,\alpha} = \Gamma_1$, then $\mathcal{B}$ is locally homogeneous.*

If $\mathcal{B}$ is holonomic or semi-holonomic, then $P = Q$ and $\Gamma_1 = \Gamma_2$. In that case, Theorem 13.3 reads as follows:

Corollary 13.4. *Let $\mathcal{B}$ be a holonomic or semi-holonomic Cosserat medium. If $\mathcal{B}$ is locally homogeneous, then the macromedium is locally homogeneous and, further, there exists a local coordinate system (x^i) around each point of $\mathcal{B}$ and a local function $\alpha(x^i)$ taking values into $\Sigma_{(1,1)}$ such that $(\Gamma_{3,-\alpha})^i_{jk} = -\alpha^i_{jk}$, or, in other words, the matrix of the Christoffel components of Γ_3 belongs to $\Sigma_{(1,1)}$. Conversely, if the macromedium is locally homogeneous and there exists a function $\alpha : \mathcal{B} \longrightarrow \Sigma_{(1,1)}$ such that $\Gamma_{3,\alpha} = \Gamma_1$, then $\mathcal{B}$ is locally homogeneous.*

13.3. The general case

Suppose that $\bar{G} = (G_1, G_2, \Sigma)$ is a Lie subgroup of $\bar{G}^2(3)$, where G_1 and G_2 are Lie subgroups of $Gl(3, \mathbb{R})$, and $\Sigma \subset B^2(3)$. We assume along this section that $(G_1, G_2, 0)$ is a subgroup of $\bar{G}$.

Theorem 13.5. *Suppose that $\mathcal{B}$ is locally homogeneous. Then the macromedium is locally homogeneous. Moreover, there exists a local coordinate system (x^i) around each point of $\mathcal{B}$ and a local function $\alpha(x^i)$ taking values into Σ such that $(\Gamma_{3,-\alpha})^i_{jk} = (\Gamma_1)^i_{jk}$, or, in other words, the connections $\Gamma_{3,-\alpha}$ and Γ_1 coincide on the domain of the coordinates (x^i). Conversely, if the macromedium is locally homogeneous and Γ_3 is a G_1-connection, then $\mathcal{B}$ is locally homogeneous.*

Proof. The proof follows the same lines that Theorems 13.1 and 13.3 taking into account that $(G_1, G_2, 0)$ is a subgroup of $\bar{G}$. □

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